

A Convenient Set of Comoving Cosmological Variables and Their Application

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ABSTRACT

A set of cosmological variables, which we shall refer to as “supercomoving variables,” are presented which are an alternative to the standard comoving variables, particularly useful for describing the gas dynamics of cosmic structure formation. For ideal gas with a ratio of specific heats $\gamma = 5/3$, the supercomoving position, velocity, and thermodynamic properties (i.e. density, temperature, and pressure) of matter are constant in time in a uniform, isotropic, adiabatically expanding universe. Expressed in terms of these supercomoving variables, the nonrelativistic, cosmological fluid conservation equations of the Newtonian approximation and the Poisson equation closely resemble their noncosmological counterparts. This makes it possible to generalize noncosmological results and techniques to address problems involving departures from uniform, adiabatic Hubble expansion in a straightforward way, for a wide range of cosmological models. These variables were initially introduced by Shandarin (1980) to describe structure formation in matter-dominated models. In this paper, we generalize supercomoving variables to models with a uniform contribution to the energy density corresponding to a nonzero cosmological constant, domain walls, cosmic strings, a nonclumping form of nonrelativistic matter (e.g. massive neutrinos in the presence of primordial density fluctuations of small wavelength), or a radiation background. Each model is characterized by the value of the density parameter Ω_0 of the non-relativistic matter component in which density fluctuation is possible, and the density parameter Ω_{X0} of the additional, nonclumping component. For each type of nonclumping background, we identify *families* within which different values of Ω_0 and Ω_{X0} lead to fluid equations and solutions in supercomoving variables which are independent of the cosmological parameters Ω_0 and Ω_{X0} . We also generalize the description to include the effects of nonadiabatic processes such as heating, radiative cooling, thermal conduction and viscosity, as well as magnetic fields in the MHD approximation.

As an illustration, we describe three familiar cosmological problems in supercomoving variables: the growth of linear density fluctuations, the nonlinear collapse of a one-dimensional plane-wave density fluctuation leading to pancake formation, and the well-known Zel’dovich approximation for extrapolating the linear growth of density fluctuations in three dimensions to the nonlinear stage.

Subject headings: cosmology: theory — dark matter — galaxies: intergalactic medium — hydrodynamics — large-scale structure of universe

1. INTRODUCTION

The modern search for an explanation for the origin of the observed structure in the universe has led to the need to solve multi-scale, highly nonlinear problems of great complexity which involve the coupling of gravitational dynamics, gas dynamics, and cosmological expansion. In what follows, we shall describe a useful tool – a change of variables — which simplifies the application of the more familiar results and techniques developed to describe a noncosmological gas in a nonexpanding background to problems in cosmology.

The description of the formation and evolution of large-scale structure in the universe is greatly simplified if we restrict our attention to systems with length scales much larger than the Schwarzschild radius of the largest mass concentrations in the universe and much smaller than the horizon, c/H , where H is the Hubble constant, with nonrelativistic peculiar motions and temperatures. In this limit, known as the Newtonian approximation, we can use the ordinary noncosmological, nonrelativistic fluid conservation equations and the Poisson equation, modified to include the effects of universal, adiabatic, Hubble expansion, to describe the evolution of the baryonic and dark matter that form the galaxies and large-scale structure we observe today.

A straightforward change of variables is sometimes introduced in which the uniform, isotropic Hubble expansion is factored out so as to obtain a set of equations that describes the evolution of departures from a uniform, isotropic, structureless universe. This is achieved by rewriting the fluid conservation equations in *comoving variables*. These variables are defined by

$$\mathbf{x} = \frac{\mathbf{r}}{a(t)}, \quad (1)$$

and

$$\rho_{\text{co-mov}} = \rho a(t)^3, \quad (2)$$

where \mathbf{r} and ρ are the proper distance and the mass density, respectively, \mathbf{x} is the comoving position, $\rho_{\text{co-mov}}$ is the comoving density, and $a(t)$ is the Robertson-Walker scale factor, whose time-evolution is described by the Friedmann equation (see eq. [4] below). In these comoving variables, in the absence of structure, the mass points are at rest and the density of ordinary, nonrelativistic matter is constant in time. For a detailed description of the Newtonian dynamics in comoving variables, see Weinberg (1972) and Peebles (1980, hereafter P80).

The fluid conservation equations in comoving variables contain several terms which distinguish them from their noncosmological counterparts. In particular, the momentum and energy equations contain “drag terms” that make the peculiar velocities and the thermal energies decay with time, even in the absence of departures from uniformity involving pressure or gravitational potential gradients to act as driving forces. This reflects the fact that universal expansion, itself, causes adiabatic cooling (i.e. PdV work is done), while the kinematics of universal expansion causes a test particle which moves relative to the universal expansion, with no peculiar forces acting on it, to appear to fall behind relative to the expanding coordinate system.

A further change of variables was introduced by Shandarin (1980, hereafter S80), referred to as “tilde variables” when modified and applied by Shapiro & collaborators (e.g. Shapiro, Struck-Marcell, & Melott 1983; Shapiro & Struck-Marcell 1985; Shapiro et al. 1996; Valinia et al. 1996). These new variables extended the concept of “comoving” to the limit in which the position, velocity, and thermodynamic properties (i.e. density, temperature, and pressure) of matter in a uniform, isotropic, adiabatically expanding, matter-dominated Friedmann universe all remain stationary. In addition, by replacing proper time by a new time variable, the so-called “drag terms” mentioned above disappeared from the momentum equation when expressed in terms of these “tilde variable.” For an ideal gas with a ratio of specific heats $\gamma = 5/3$, the “drag term” in the energy equation disappeared, as well. For such a gas, the complete set of three fluid

conservation equations, for matter, momentum, and energy, when expressed in tilde variables, are identical to the standard, noncosmological fluid equations. Henceforth, we shall refer to these tilde variables as “supercomoving variables.” instead, in order to give a more descriptive name.

The use of supercomoving variables makes it possible to take the known solutions of non-cosmological problems and generalize them in a straight-forward way to cosmological situations. An example of this is provided by Voit (1996), who used supercomoving variables to address the problem of the evolution of intergalactic blast waves in a matter-dominated universe. The similarity of the fluid equations in supercomoving and noncosmological variables also means that analytical methods and numerical simulation algorithms which are suitable for studying non-cosmological problems can often be applied to cosmological ones with essentially no modification. Examples of the latter include the 1D, Lagrangian hydrodynamics method developed by Shapiro et al. (1983) and Shapiro & Struck-Marcell (1985) to study cosmological pancakes and the new anisotropic version of Smoothed Particle Hydrodynamics (SPH), called Adaptive SPH (ASPH), developed recently by Shapiro et al. (1996) to study both cosmological and noncosmological gas dynamics by numerical simulation in 2D and 3D.

Shandarin (S80) also showed that, with an appropriate normalization of the scale factor $a(t)$, it was possible to eliminate all dependences, implicit or explicit, upon the density parameter Ω_0 . All matter-dominated cosmological models can be grouped into three distinct families, $\Omega_0 < 1$, $\Omega_0 = 1$, and $\Omega_0 > 1$, and within each family, the fluid conservation equations and their solutions, and the solutions of the Friedmann equation, are identical. There is no distinction in supercomoving variables between a model with $\Omega_0 = 0.2$ and $\Omega_0 = 0.99$, for instance.

The supercomoving variables introduced by Shandarin (S80) were restricted to matter-dominated models (i.e. models in which the mean energy density is dominated by nonrelativistic matter [baryonic and dark], so $\Omega_{X0} = 0$). There are several reasons, however, for considering cosmological models that contain more than just the nonrelativistic component. For the particular case of a nonzero cosmological constant, these reasons have been summarized by Carroll, Press, & Turner (1992), Ostriker & Steinhardt (1995), and Krauss & Turner (1995). The case of a more general, smooth background component has been discussed recently by Turner & White (1997). Some of the arguments are as follows. Dynamical estimates of the density parameter Ω_0 have consistently given values smaller than unity (see, for example, Peebles 1993 and references therein) (see, however, Dekel et al. 1993, and references therein for an exception to this trend), in contradiction with the most common inflationary scenario (Guth 1981; Linde 1982; Albrecht & Steinhardt 1982) which requires $\Omega_0 = 1$. (We note, however, that inflationary models with $\Omega_0 < 1$ are now being considered [Lyth & Steward 1990; Ratra & Peebles 1994].) To reconcile the dynamical estimates of Ω_0 with standard inflation, we must postulate the existence of a second component which must be uniform in space in order to escape detection by dynamical methods (hence, we are not referring to the dark matter component in galaxy clusters and galactic halos; this component is already included in dynamical estimates of Ω_0). The presence of this uniform component such as domain walls or a cosmological constant also increases the age of the universe for a given value of H_0 . This helps to reconcile current estimates of H_0 , which favor $H_0 \approx 60 - 80 \text{ km s}^{-1} \text{ Mpc}^{-1}$ (Riess, Kirshner, & Press 1995; Freedman 1996; Giovanelli et al 1996), with lower limits to the age of the universe based on estimates of globular clusters ages, the well-known age problem. With the detection of temperature anisotropy by the COBE satellite DMR experiment, it has become possible to fix the amplitude and constrain the wavelength dependence of the primordial density fluctuations responsible for large-scale structure (LSS) formation, at least at the superhorizon scale at recombination to which the COBE measurement is sensitive (see Bunn & White 1997, and references therein). The much-studied, standard, flat, matter-dominated Cold Dark Matter (CDM) model for the origin of galaxies and LSS is, however, incompatible with these COBE constraints since it tends to exaggerate the structure on the smaller scales measured by the statistical clustering properties of galaxies and LSS in comparison with galaxy data (see Ostriker 1993 [§3.2], and references therein; Ostriker & Steinhardt 1995, and references therein). The flat CDM model can be reconciled with these COBE results, however, if there is a uniform

background component like a cosmological constant which dominates at late times over the nonrelativistic matter component.

The main goal of this paper, therefore, is to generalize the original supercomoving variables of Shandarin, which are limited to matter-dominated models, to include models containing an additional, uniform component. In the process, we will demonstrate that different cosmological models corresponding to different combinations of the fundamental cosmological parameters can be grouped into “families” in which the evolution of physical quantities is the same when expressed in supercomoving variables. We will follow Shapiro et al. (1983) and Shapiro & Struck-Marcell (1985) in the inclusion of non-adiabatic processes like external heating, radiative cooling, and thermal conduction, and extend their description to include viscosity, vorticity, and the effects of magnetic fields in the MHD approximation.

The remainder of the paper is organized as follows. In §2, we present the fluid equations in noncosmological variables. In §3, we define the supercomoving variables. In §4, we derive the supercomoving form of the fluid equations. In §5, we present the supercomoving form of the collisionless Boltzman equation. In §6, we derive the cosmic energy equation in supercomoving variables. In §7, we solve the Friedmann equation for the various models considered in this paper, and derive the conditions required in order to have families of solutions which are independent of the particular values of the cosmological parameters for different combinations of these parameters. In §8, we present the solutions of the linear perturbation equations for the various models considered. In §9, we discuss exact, nonlinear solutions for the collapse of plane-wave density fluctuations, prior to shock and caustic formation. In §10, we present the supercomoving form of the Zel’dovich approximation. Summary and conclusions are presented in §11. Our Appendix A discusses additional physical processes, including viscosity, heating, cooling, thermal conduction, vorticity, and magnetic fields in the MHD approximation.

2. BASIC EQUATIONS IN NONCOSMOLOGICAL COORDINATES

2.1. The Cosmological Background

The evolution of the cosmological background is determined by the relative contribution by its constituent components to the mean mass-energy density of the universe, as it appears in the Einstein field equations. For an isotropic, homogeneous universe with a given composition, this evolution is expressed in terms of the Friedmann-Robertson-Walker metric with a cosmic scale factor $a(t)$ whose time-evolution is given by the Friedmann equation.

In this paper, we shall consider two-component models. The first component, called the *nonrelativistic (NR) component*, comprises all forms of matter, luminous or dark, baryonic or non-baryonic, that can cluster under the action of gravity, and whose *mean* energy density varies with time as $a(t)^{-3}$. The second component, which we shall refer to as the *X component*, does not clump and, instead, has a *uniform* energy density ρ_X which varies with expansion according to

$$\rho_X(t) \propto a(t)^{-n}, \quad (3)$$

where n is a non-negative constant. This equation was first introduced by Fry (1985) (see also Charlton & Turner 1987; Silveira & Waga 1994; Martel 1995; Dodelson, Gates, & Turner 1996; Turner & White 1997). By appropriate choice of n , we recover models with a nonzero cosmological constant ($n = 0$), domain walls ($n = 1$), string networks ($n = 2$), vacuum stress ($n = 3$), or a radiation background ($n = 4$). In this paper, we shall derive a form of the supercomoving variables applicable to all of these various models. Notice that models with massive, nonrelativistic neutrinos are also described by equation (3) with $n = 3$, as long as the length scale of the problem studied is much shorter than the damping scale of the neutrinos. For $n \neq 3$, the

X component has a relativistic pressure $P_X = \nu \rho_X c^2$, where c is the speed of light and $\nu = n/3 - 1$, which enters in the Newtonian form of the Poisson equation as an additional source term (see eq. [8] below).

The time-evolution of the scale factor $a(t)$ is described by the Friedmann equation. For the two-component models considered in this paper, this equation takes the form

$$\left(\frac{\dot{a}}{a}\right)^2 = H(t)^2 = H_0^2 \left[(1 - \Omega_0 - \Omega_{X0}) \left(\frac{a}{a_0}\right)^{-2} + \Omega_0 \left(\frac{a}{a_0}\right)^{-3} + \Omega_{X0} \left(\frac{a}{a_0}\right)^{-n} \right], \quad (4)$$

(Martel 1995) where we use subscripts 0 to indicate the present value of time-dependent quantities. The first term in the right hand side is the curvature term $-k/a^2$, where $k = -H_0^2 a_0^2 (1 - \Omega_0 - \Omega_{X0})$. The other two terms represent the nonrelativistic and X component, respectively. The density parameters Ω and Ω_X are defined by $\Omega = \bar{\rho}/\rho_c$ and $\Omega_X = \rho_X/\rho_c$, where $\bar{\rho}$ and $\rho_X = \bar{\rho}_X$ are the mean density of the nonrelativistic and X component, respectively, and $\rho_c \equiv 3H^2/8\pi G$ is the critical density, defined as the mean density of a flat universe, *in the absence of the X component*.

2.2. Fluid Conservation Equations in the Newtonian Approximation

The dynamics of a self-gravitating fluid in the Newtonian approximation is described by the basic hydrodynamical equations for the conservation of mass, momentum, and energy, coupled to the Poisson equation and the equation of state. The Newtonian approximation and its validity in cosmology is discussed by P80. It is valid in the limit in which all length scales are much smaller than the horizon size, $c/H(t)$, and much larger than the Schwarzschild radii of objects within the region of study, and temperatures, equations of state, and peculiar velocities are nonrelativistic. In proper coordinates and in the absence of viscosity or dissipative processes like thermal conduction or radiative cooling, the continuity, momentum, and energy equations, and the equation of state take the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (5)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \Phi - \frac{\nabla p}{\rho}, \quad (6)$$

$$\frac{\partial \varepsilon}{\partial t} + \mathbf{u} \cdot \nabla \varepsilon = -\frac{p}{\rho} \nabla \cdot \mathbf{u}, \quad (7)$$

$$p = (\gamma - 1) \varepsilon \rho, \quad (8)$$

where ρ is the matter density, ε is the specific internal energy, p is the gas pressure, \mathbf{u} is the velocity, Φ is the gravitational potential, and γ is the ratio of specific heats. These equations also assume that there is no interaction besides gravity between the nonrelativistic component and the X component. In Appendix A, we generalize these equations to include additional physical processes, such as heating, cooling, thermal conduction, viscosity, and magnetic fields.

2.3. Gravitational Potential in the Newtonian Approximation: Modified Poisson Equation

The Newtonian form of the Poisson equation must be modified to account for the presence of the X component. To derive the Poisson equation, we consider the acceleration g^α between two freely-falling particles separated by a proper distance ξ^α . In the *small-scale limit*, where ξ^α is much smaller than the radius

of curvature of the universe, the space-time is locally Minkowskian, and the equation for the acceleration reduces to

$$g^\alpha = R^\alpha_{00\beta} \xi^\beta, \quad (9)$$

(Peebles 1993, p. 268) where R^i_{jkl} is the Riemann-Christoffel curvature tensor, and where we sum over repeated indicies. By taking the divergence of equation (9), we get

$$\nabla^2 \Phi \equiv -\nabla \cdot \mathbf{g} = 4\pi G T^{\alpha\alpha}, \quad (10)$$

where \mathbf{T} is the stress-energy tensor. For the models we consider, the total stress-energy tensor in the fluid rest frame is given by

$$\mathbf{T} = \begin{bmatrix} \rho + \rho_X & 0 & 0 & 0 \\ 0 & p_X/c^2 & 0 & 0 \\ 0 & 0 & p_X/c^2 & 0 \\ 0 & 0 & 0 & p_X/c^2 \end{bmatrix}, \quad (11)$$

where ρ_X and p_X are the energy density and pressure of the X component, respectively, and c is the speed of light. In equation (11), we neglected the nonrelativistic pressure term p/c^2 , which is smaller than ρ by a factor of order $(c_s/c)^2$, where c_s is the sound speed. The evolution of the density ρ_X of the X component is given by

$$\frac{d}{da}(\rho_X a^3) = -\frac{3p_X a^2}{c^2} \quad (12)$$

(Weinberg 1972, p. 472). By combining equations (3) and (12), then yield

$$\frac{p_X}{c^2} = \left(\frac{n}{3} - 1\right) \rho_X. \quad (13)$$

Equations (10), (11), and (13) reduces to

$$\nabla^2 \Phi = 4\pi G \left[\rho + (n-2)\rho_X \right]. \quad (14)$$

This is the final form of the Poisson equation in the presence of the X component. Notice that the contribution of the X component to the gravitational source term is negative for $n < 2$. This implies that the gravitational force resulting from the presence of domain walls or a cosmological constant is repulsive.

3. SUPERCOMOVING VARIABLES

One of the most desirable features of the comoving and supercomoving variables is that in these variables, in the absence of structure, the density is constant in time and mass elements are at rest. In supercomoving variables, a gas with ratio of specific heats $\gamma = 5/3$ also has its thermodynamic variables remain constant in the absence of structure. We preserve these two properties when we generalize supercomoving variables to the cases involving an additional nonclumping background component. by making the following definitions,

$$\tilde{\mathbf{r}} = \frac{\mathbf{r}}{ar_*}, \quad (15)$$

$$\tilde{\rho} = \frac{a^3 \rho}{\rho_*}, \quad (16)$$

$$\tilde{\mathbf{v}} = \frac{a\mathbf{v}}{v_*}, \quad (17)$$

$$d\tilde{t} = \frac{dt}{a^2 t_*}, \quad (18)$$

$$\tilde{\phi} = \frac{a^2 \phi}{\phi_*}, \quad (19)$$

$$\tilde{p} = \frac{a^5 p}{p_*}, \quad (20)$$

$$\tilde{\varepsilon} = \frac{a^2 \varepsilon}{\varepsilon_*}, \quad (21)$$

where the peculiar velocity \mathbf{v} is defined by

$$\mathbf{v} = \mathbf{u} - H\mathbf{r}, \quad (22)$$

where $H \equiv a^{-1}da/dt$ is the Hubble constant, and the peculiar gravitational potential ϕ is related to the Eulerian gravitational potential Φ by

$$\Phi = \frac{2\pi G \bar{\rho} r^2}{3} + \frac{2(n-2)\pi G \bar{\rho}_X r^2}{3} + \phi = \frac{\Omega_0 H_0^2}{4} \left(\frac{a_0}{a}\right)^3 r^2 + \frac{\Omega_{X0} H_0^2 (n-2)}{4} \left(\frac{a_0}{a}\right)^n r^2 + \phi. \quad (23)$$

This is a standard transformation, except for the second term which we introduce to take into account the presence of the X component. In equations (15)–(21), the quantities r_* , ρ_* , \mathbf{v}_* , t_* , ϕ_* , p_* , and ε_* are fiducial values. Only r_* is independent. The other fiducial quantities are defined by

$$\rho_* \equiv \bar{\rho}_0 = \frac{3H_0^2 \Omega_0}{8\pi G}, \quad (24)$$

$$t_* \equiv \frac{2}{H_0} \left(\frac{f_n}{\Omega_0 a_0^3}\right)^{1/2}, \quad (25)$$

$$v_* \equiv \frac{r_*}{t_*}, \quad (26)$$

$$\phi_* \equiv r_*^2/t_*^2 = v_*^2, \quad (27)$$

$$p_* \equiv \rho_* r_*^2/t_*^2 = \rho_* v_*^2, \quad (28)$$

$$\varepsilon_* \equiv p_*/\rho_* = v_*^2, \quad (29)$$

where $\bar{\rho}_0$ is the cosmic mean matter density at the present epoch and the parameter f_n is defined by

$$f_n = \begin{cases} 1, & n \neq 3; \\ \frac{\Omega_0}{\Omega_0 + \Omega_{X0}}, & n = 3. \end{cases} \quad (30)$$

For any particular cosmological model, we can eliminate $a(t)$ using the Friedmann equation, and integrate equation (18) to get an explicit relation between t and \tilde{t} . For future reference, equation (18), together with the Friedman equation (4), implies the following expressions for the Hubble constant and the first and second derivatives of the scale factor,

$$H(t) = \frac{1}{a^3 t_*} \frac{da}{d\tilde{t}} \equiv \frac{1}{a^2 t_*} \mathcal{H}, \quad (31)$$

$$\left(\frac{da}{d\tilde{t}}\right)^2 = t_*^2 H_0^2 a_0^6 \left[(1 - \Omega_0 - \Omega_{X0}) \left(\frac{a}{a_0}\right)^4 + \Omega_0 \left(\frac{a}{a_0}\right)^3 + \Omega_{X0} \left(\frac{a}{a_0}\right)^{6-n} \right], \quad (32)$$

$$\frac{d^2 a}{d\tilde{t}^2} = t_*^2 H_0^2 a_0^5 \left[2(1 - \Omega_0 - \Omega_{X0}) \left(\frac{a}{a_0} \right)^3 + \frac{3\Omega_0}{2} \left(\frac{a}{a_0} \right)^2 + \left(3 - \frac{n}{2} \right) \Omega_{X0} \left(\frac{a}{a_0} \right)^{5-n} \right], \quad (33)$$

where equation (31) defines the quantity $\mathcal{H} = a^{-1} da/d\tilde{t}$, the supercomoving Hubble constant. By combining equations (31) and (32), we can eliminate $da/d\tilde{t}$ and express \mathcal{H} as a function of a .

4. SUPERCOMOVING FLUID EQUATIONS

4.1. Transformation of Derivatives

To convert the hydrodynamical equations (5)–(8) and (14) to supercomoving variables, we need to express the derivatives relative to \mathbf{r} at fixed t and \tilde{t} at fixed \mathbf{r} as functions of the derivatives relative to $\tilde{\mathbf{r}}$ at fixed \tilde{t} and \tilde{t} at fixed $\tilde{\mathbf{r}}$, using equations (15) and (18). This yields

$$\left(\frac{\partial f}{\partial t} \right)_{\mathbf{r}} = \frac{1}{a^2 t_*} \left[\left(\frac{\partial f}{\partial \tilde{t}} \right)_{\tilde{\mathbf{r}}} - \mathcal{H} \tilde{\mathbf{r}} \cdot (\tilde{\nabla} f)_{\tilde{t}} \right], \quad (34)$$

$$(\nabla f)_t = \frac{1}{a r_*} (\tilde{\nabla} f)_{\tilde{t}}, \quad (35)$$

where $\tilde{\nabla}$ is the gradient relative to $\tilde{\mathbf{r}}$.

4.2. The Continuity Equation

To derive the form of the continuity equation in supercomoving variables, we substitute equations (15), (16), (17), (22), (24), (26), (34), and (35) into equation (5). After some algebra, we get

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\rho} \tilde{\mathbf{v}}) = 0. \quad (36)$$

This equation has exactly the same form as equation (5).

4.3. The Momentum Equation

We substitute equations (15), (16), (17), (19), (20), (22)–(28), (34), and (35) into equation (6), and eliminate the terms in $da/d\tilde{t}$ and $d^2 a/d\tilde{t}^2$ using equations (32) and (33). We get, after some algebra,

$$\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + (\tilde{\mathbf{v}} \cdot \tilde{\nabla}) \tilde{\mathbf{v}} = - \frac{\tilde{\nabla} \tilde{p}}{\tilde{\rho}} - \tilde{\nabla} \tilde{\phi}. \quad (37)$$

Again, we have obtained an equation that has the same form in supercomoving variables as in noncosmological variables. The corresponding equation in comoving variables contains an additional drag term.

4.4. The Poisson Equation

We substitute equations (16), (19), (23), (24), (25), (27), and (35) into equation (14), and get

$$\tilde{\nabla}^2 \tilde{\phi} = 6a f_n \left(\frac{\tilde{\rho}}{\tilde{\bar{\rho}}} - 1 \right), \quad (38)$$

where we used $\tilde{\bar{\rho}} \equiv a^3 \bar{\rho} / \rho_* = a^3 \bar{\rho} / \bar{\rho}_0 = a_0^3$ (see eqs. [16] and [23]) to eliminate a_0 in equation (24). For $n = 3$, we can rewrite equation (38) as

$$\tilde{\nabla}^2 \tilde{\phi} = 6a \left(\frac{\tilde{\rho}_{\text{tot}}}{\tilde{\bar{\rho}}_{\text{tot}}} - 1 \right), \quad (39)$$

where $\rho_{\text{tot}} = \rho + \bar{\rho}_X$ is the total density, *including the X component*. The reason for treating the case $n = 3$ differently will be explained in §7.3.3 below.

4.5. The Equation of State

We substitute equations (16), (20), (21), (24), (28), and (29) into equation (8). The equation of state becomes

$$\tilde{p} = (\gamma - 1) \tilde{\rho} \tilde{\varepsilon}, \quad (40)$$

which has the same form as equation (8).

4.6. The Energy Equation

We substitute equations (15), (16), (17), (20), (21), (22), (24), (26), (28), (29), (34), and (35) into equation (7). The energy equation becomes

$$\frac{\partial \tilde{\varepsilon}}{\partial \tilde{t}} + \tilde{\mathbf{v}} \cdot \tilde{\nabla} \tilde{\varepsilon} + \mathcal{H}(3\gamma - 5) \tilde{\varepsilon} = -\frac{\tilde{p}}{\tilde{\rho}} \tilde{\nabla} \cdot \tilde{\mathbf{v}}. \quad (41)$$

This equation contains a drag term, which vanishes only for $\gamma = 5/3$. We can derive the evolution of the mean specific internal energy $\tilde{\varepsilon}$ by considering a universe with no fluctuation. In this case, $\tilde{\mathbf{v}} = 0$ and $\partial/\partial \tilde{t} = d/d\tilde{t}$. Equation (41) then becomes

$$\frac{d\tilde{\varepsilon}}{d\tilde{t}} + \mathcal{H}(3\gamma - 5) \tilde{\varepsilon} = 0. \quad (42)$$

The solution is

$$\tilde{\varepsilon}(t) = \tilde{\varepsilon}_0 \left(\frac{a}{a_0} \right)^{5-3\gamma}. \quad (43)$$

Hence, the mean specific internal energy decreases with expansion for $\gamma > 5/3$, and increases for $\gamma < 5/3$. Since the mean mass density does not vary with expansion in supercomoving variables, the equation of state gives

$$\tilde{p}(t) = \tilde{p}_0 \left(\frac{a}{a_0} \right)^{5-3\gamma}. \quad (44)$$

The reason for the absence of drag term in equation (41) for the case $\gamma = 5/3$ only is easily understood. The momentum equation (37) contains no drag term. From this equation, we can derive an equation for the evolution of the specific kinetic energy $\tilde{v}^2/2$, which contains no drag term either. In the case of $\gamma = 5/3$, the internal energy of the gas is nothing more than the microscopic specific kinetic energy of the particles that compose the gas. Hence, because of the absence of drag term, the mean specific internal energy remains constant as the universe expands. But if γ is not $5/3$, then there is more to the internal energy of the gas than just the specific kinetic energy of the particles. For instance, for diatomic gases with $\gamma = 7/5$, some of the internal energy is in the form of molecular rotation. Hence, the drag term in the energy equation (41) represents the component of the internal energy that is not associated with the microscopic motion of the atoms and molecules in the gas.

5. THE SUPERCOMOVING COLLISIONLESS BOLTZMANN EQUATION

The equations derived in §4 describe the evolution of a collisional gas. These equations cannot be used for describing the evolution of a collisionless system of particles, because the velocity \mathbf{u} is a multivalued quantity. The proper way to describe such system is in terms of a distribution function $f(\mathbf{r}, \mathbf{u}, t)$ which is defined as the number density of particles in phase space. The evolution of the distribution function is described by the collisionless Boltzmann equation, also known as the Vlasov equation. In noncosmological variables, this equation takes the form

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f - \nabla \Phi \cdot \nabla_{\mathbf{u}} f = 0, \quad (45)$$

where the gravitational potential is obtained by solving the Poisson equation, which, for collisionless systems, is given by

$$\nabla^2 \Phi = 4\pi G \left[\int f d^3 u + \rho_b \right], \quad (46)$$

where we included the baryon density ρ_b to take into account systems in which the nonrelativistic component is a mixture of a collisionless and a collisional (gaseous) component. After switching to supercomoving variables, the Boltzmann equation becomes

$$\frac{\partial \tilde{f}}{\partial \tilde{t}} + \tilde{\mathbf{v}} \cdot \tilde{\nabla} \tilde{f} - \tilde{\nabla} \tilde{\phi} \cdot \tilde{\nabla}_{\tilde{\mathbf{v}}} \tilde{f} = 0, \quad (47)$$

where $\tilde{f}(\tilde{\mathbf{r}}, \tilde{\mathbf{v}}, \tilde{t}) \equiv f(\mathbf{r}, \mathbf{u}, t)$, and the Poisson equation becomes

$$\nabla^2 \tilde{\phi} = \begin{cases} 6a \left(\frac{\int \tilde{f} d^3 \tilde{\mathbf{v}} + \tilde{\rho}_b}{\langle \int \tilde{f} d^3 \tilde{\mathbf{v}} \rangle + \tilde{\rho}_b} - 1 \right), & n \neq 3; \\ 6a \left(\frac{\int \tilde{f} d^3 \tilde{\mathbf{v}} + \tilde{\rho}_X + \tilde{\rho}_b}{\langle \int \tilde{f} d^3 \tilde{\mathbf{v}} \rangle + \tilde{\rho}_X + \tilde{\rho}_b} - 1 \right), & n = 3; \end{cases} \quad (48)$$

where $\langle \rangle$ indicates space-averaging.

6. THE COSMIC ENERGY EQUATION

Using the fluid equations derived in the previous sections, we can derive an equation for the evolution of the total energy. Our derivation follows the same lines as the one presented in P80 for the cosmic energy

equation in comoving variables. Taking the dot product of $\tilde{\rho}\tilde{\mathbf{v}}$ with the momentum equation (37), we get

$$\frac{1}{2} \frac{\partial(\tilde{\rho}\tilde{v}^2)}{\partial\tilde{t}} - \frac{\tilde{v}^2}{2} \left[\frac{\partial\tilde{\rho}}{\partial\tilde{t}} + \tilde{\nabla} \cdot (\tilde{\rho}\tilde{\mathbf{v}}) \right] = -\tilde{\nabla} \cdot \left(\tilde{\rho}\tilde{\mathbf{v}} + \tilde{\rho}\tilde{\phi}\tilde{\mathbf{v}} + \frac{\tilde{\rho}\tilde{v}^2\tilde{\mathbf{v}}}{2} \right) + \tilde{\rho}\tilde{\nabla} \cdot \tilde{\mathbf{v}} + \tilde{\phi}\tilde{\nabla} \cdot (\tilde{\rho}\tilde{\mathbf{v}}). \quad (49)$$

Using the continuity equation (36), we cancel the term in square brackets, and eliminate $\tilde{\nabla} \cdot (\tilde{\rho}\tilde{\mathbf{v}})$ in the last term. We also use the energy equation (41) to eliminate the $\tilde{\rho}\tilde{\nabla} \cdot \tilde{\mathbf{v}}$ term. After some additional vector manipulation, we get

$$\frac{1}{2} \frac{\partial(\tilde{\rho}\tilde{v}^2)}{\partial\tilde{t}} = -\tilde{\nabla} \cdot \left(\tilde{\rho}\tilde{\mathbf{v}} + \tilde{\rho}\tilde{\phi}\tilde{\mathbf{v}} + \frac{\tilde{\rho}\tilde{v}^2\tilde{\mathbf{v}}}{2} + \tilde{\rho}\tilde{\varepsilon}\tilde{\mathbf{v}} \right) - \frac{\partial(\tilde{\rho}\tilde{\varepsilon})}{\partial\tilde{t}} + \tilde{\varepsilon} \left[\frac{\partial\tilde{\rho}}{\partial\tilde{t}} + \tilde{\nabla} \cdot (\tilde{\rho}\tilde{\mathbf{v}}) \right] - \mathcal{H}(3\gamma - 5)\tilde{\rho}\tilde{\varepsilon} - \tilde{\phi} \frac{\partial\tilde{\rho}}{\partial\tilde{t}}. \quad (50)$$

Again, we cancel the term in square brackets using the continuity equation. We then integrate equation (50) over all space. For a system with finite extent, the divergence term vanishes. We now introduce the following definitions for the total kinetic energy and total thermal energy of the system:

$$\tilde{T} \equiv \frac{1}{2} \int \rho\tilde{v}^2 d^3\tilde{r}, \quad (51)$$

$$\tilde{E} \equiv \int \tilde{\rho}\tilde{\varepsilon} d^3\tilde{r}. \quad (52)$$

With these definitions, equation (50) reduces to

$$\frac{d\tilde{T}}{d\tilde{t}} = -\frac{d\tilde{E}}{d\tilde{t}} - \mathcal{H}(3\gamma - 5)\tilde{E} - \int \tilde{\phi} \frac{\partial(\tilde{\rho} - \tilde{\bar{\rho}})}{\partial\tilde{t}} d^3\tilde{r}. \quad (53)$$

Notice that integration over all space has transformed the partial time derivatives into total time derivatives, and that we introduced the constant mean density $\tilde{\bar{\rho}}$ in the last term. To compute the last term in equation (53), we first solve the Poisson equation (38), and get

$$\tilde{\phi}(\tilde{\mathbf{r}}) = -\frac{3af_n}{2\pi\tilde{\bar{\rho}}} \int \frac{\tilde{\rho}(\tilde{\mathbf{r}}') - \tilde{\bar{\rho}}}{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}'|} d^3\tilde{r}'. \quad (54)$$

We can easily show that this is indeed the correct solution by directly comparing equations (38) and (54) to their Eulerian counterparts. With this solution, the last term in equation (53) becomes

$$-\int \tilde{\phi} \frac{\partial(\tilde{\rho} - \tilde{\bar{\rho}})}{\partial\tilde{t}} d^3\tilde{r} = -\frac{a}{2} \frac{d}{d\tilde{t}} \left[\frac{\int \tilde{\phi}(\tilde{\mathbf{r}}) [\tilde{\rho}(\tilde{\mathbf{r}}) - \tilde{\bar{\rho}}] d^3\tilde{r}}{a} \right]. \quad (55)$$

We then introduce the following definition for the gravitational potential energy of the system:

$$\tilde{U} \equiv \frac{1}{2} \int \tilde{\phi}(\tilde{\rho} - \tilde{\bar{\rho}}) d^3\tilde{r}. \quad (56)$$

Equation (53) reduces to

$$\frac{d}{d\tilde{t}} (\tilde{T} + \tilde{E} + \tilde{U}) = \mathcal{H} \left[(5 - 3\gamma)\tilde{E} + \tilde{U} \right]. \quad (57)$$

This is the final form of the cosmic energy equation in supercomoving variables. The corresponding equation in comoving variables (see P80, eq. [24.19]) is significantly more complicated.

7. SOLUTIONS OF THE FRIEDMANN EQUATION

7.1. Cases with Elementary Solutions

The evolution of the cosmological background is described by the Friedmann equation (4), written in supercomoving variables in equation (32). For convenience, we rewrite equation (32) as follows,

$$\left(\frac{dx}{d\tau}\right)^2 = \frac{4(1 - \Omega_0 - \Omega_{X0})}{\Omega_0}x^4 + 4x^3 + \frac{4\Omega_{X0}}{\Omega_0}x^{6-n}, \quad (58)$$

where

$$x = a_0^{-1}a, \quad (59)$$

$$\tau = (a_0 f_n)^{1/2} \tilde{t}. \quad (60)$$

The solution of this equation is

$$\tau = \frac{1}{2} \int dx \left(\frac{1 - \Omega_0 - \Omega_{X0}}{\Omega_0} x^4 + x^3 + \frac{\Omega_{X0}}{\Omega_0} x^{6-n} \right)^{-1/2}. \quad (61)$$

This integral has no elementary solution for arbitrary values of the parameters Ω_0 , Ω_{X0} , and n . There are, however, several physically interesting cases for which elementary solutions exist. We can identify the cases that have or might have an elementary solution by studying the properties of equation (61).

First, all known cases which are physically relevant have $0 \leq n \leq 4$, with n integer. Hence, we can rewrite equation (61) as

$$\tau = \frac{1}{2} \int \frac{dx}{x} P(x)^{-1/2}, \quad (62)$$

where

$$P(x) = \frac{1 - \Omega_0 - \Omega_{X0}}{\Omega_0} x^2 + x + \frac{\Omega_{X0}}{\Omega_0} x^{4-n} \quad (63)$$

is a polynomial of degree 4 or less. If $n = 2, 3$, or 4 , $P(x)$ is a second degree polynomial, and equation (62) always has an elementary solution. If $n = 1$ (domain walls) or $n = 0$ (cosmological constant), then $P(x)$ is a polynomial of degree 3 or 4, the integral in equation (62) is elliptic, and does not have an elementary solution in general. There are special cases, however. Suppose that $P(x)$ has a double root at some particular value $x = x_{st}$. We can then rewrite $P(x)$ as

$$P(x) = (x - x_{st})^2 Q(x), \quad (64)$$

where $Q(x)$ is a linear polynomial if $n = 1$ or a quadratic polynomial if $n = 0$. Substituting equation (64) into equation (62), we get

$$\tau = \frac{1}{2} \int \frac{dx}{x|x - x_{st}|} Q(x)^{-1/2}, \quad (65)$$

for which there is always an elementary solution. The necessary condition for the polynomial $P(x)$ to have a double root is that $P(x)$ and its first derivative both vanish at $x = x_{st}$, which becomes

$$\left(\frac{1 - \Omega_0 - \Omega_{X0}}{\Omega_0} \right) x_{st}^2 + x_{st} + \frac{\Omega_{X0}}{\Omega_0} x_{st}^{4-n} = 0, \quad (66)$$

$$2 \left(\frac{1 - \Omega_0 - \Omega_{X0}}{\Omega_0} \right) x_{st} + 1 + (4 - n) \frac{\Omega_{X0}}{\Omega_0} x_{st}^{3-n} = 0. \quad (67)$$

We can eliminate x_{st} from these equations to get a condition relating Ω_0 and Ω_{X0} . We get, after some algebra,

$$1 - \Omega_0 - \Omega_{X0} = \begin{cases} -3 \left(\frac{\Omega_{X0} \Omega_0^2}{4} \right)^{1/3}, & n = 0; \\ -2(\Omega_{X0} \Omega_0)^{1/2}, & n = 1. \end{cases} \quad (68)$$

The value of x_{st} is then given by

$$x_{\text{st}} = \begin{cases} \left(\frac{\Omega_0}{2\Omega_{X0}} \right)^{1/3}, & n = 0; \\ \left(\frac{\Omega_0}{\Omega_{X0}} \right)^{1/2}, & n = 1. \end{cases} \quad (69)$$

The physical meaning of x_{st} is easily understood. By combining equations (58), (66), and (67), we can show that \dot{a} and \ddot{a} both vanish at $x = x_{\text{st}}$; hence the universe is *static* at $a = a_0 x_{\text{st}}$. The case $n = 0$ is known as the Einstein static model. In this model, the repulsive force caused by the cosmological constant exactly cancels the gravitational force. As we see, the possibility of a repulsive force canceling gravity also exists for domain walls (this is also shown by eq. [14], in which the X component acts as a negative density for $n < 2$ models). Since the universe is clearly expanding at present, x must be smaller than the equilibrium value x_{st} . Equations (58) and (68) then describe a universe which decelerates and asymptotically approaches a static universe as x gets closer to x_{st} .

7.2. Families of Solutions

Equation (61) (or [32]) relates the cosmic scale factor a to our supercomoving time variable \tilde{t} in any particular model. According to this equation, models with different values of Ω_0 and Ω_{X0} can often be grouped into “families” within which the dependence of a on \tilde{t} is the same, as follows.

There is a basic difference which occurs between those cases for which $P(x)$ defined by equation (63) reduces to only two terms and those cases for which $P(x)$ has three terms. Let us call the former case, category 1, while the latter is category 2. For cases in category 1, it is always possible to find solutions with no explicit dependences upon the cosmological parameters Ω_0 and Ω_{X0} . If $P(x)$ contains only two terms, the change of variable $x = by$, with an appropriate choice for b , will make the coefficients of these two terms identical. We can then move that coefficient to the left hand side of equation (62) and eliminate it by rescaling τ . Cases for which $P(x)$ has only two terms include matter-dominated models ($\Omega_{X0} = 0$, as described in S80), any zero-curvature model ($1 - \Omega_0 - \Omega_{X0} = 0$) regardless of the presence of an additional uniform background component, models with cosmic strings ($n = 2$), and models with a nonrelativistic uniform component ($n = 3$). In each of these cases in category 1, the solution for $a(\tilde{t})$ as a function of \tilde{t} reduces to a finite number of functions which describe the infinite number of combinations of values for Ω_0 and Ω_{X0} which can occur in each case. We present these solutions in §7.3. In all other cases, $P(x)$ contains 3 terms (category 2), and there are an infinite number of solutions for the dependence of $a(\tilde{t})$ on \tilde{t} , but we can still identify combinations of Ω_0 and Ω_{X0} which share a common solution. To find these combinations, we set $x = (\Omega_0/\Omega_{X0})^{1/(3-n)}y$ in equation (63), which makes the coefficients of the last two terms identical. We get

$$P(y) = \left(\frac{\Omega_0}{\Omega_{X0}} \right)^{1/(3-n)} \left[\frac{(1 - \Omega_0 - \Omega_{X0})\Omega_0^{(n-2)/(3-n)}}{\Omega_{X0}^{1/(3-n)}} y^2 + y + y^{4-n} \right]. \quad (70)$$

As long as the following condition is met:

$$\kappa \equiv \frac{(1 - \Omega_0 - \Omega_{X0})\Omega_0^{(n-2)/(3-n)}}{\Omega_{X0}^{1/(3-n)}} = \text{const}, \quad (71)$$

the solution will not depend upon Ω_0 and Ω_{X0} . For any combination of Ω_0 and Ω_{X0} which satisfies equation (71), there is a single dependence of $a(\tilde{t})$ on \tilde{t} , which is different for different values of κ . There are, therefore, an infinite number of such cases, corresponding to the infinite range of values of κ . The particular cases with $n = 0$ or 1 for which closed-form, analytical solutions exist (eq. [68]) do satisfy the condition in equation (71).

Henceforth, we shall refer to the combination of values of Ω_0 , Ω_{X0} , and n which share a common solution for the dependence of $a(\tilde{t})$ on \tilde{t} as “families.” The matter-dominated cases ($\Omega_{X0} = 0$) considered by Shandarin (S80) belong to category 1 of the cases described above. As shown by Shandarin, these models which are parametrized by Ω_0 are, in supercomoving variables, fully described by three families, closed, flat, and open, and within each family, the dependence of $a(\tilde{t})$ on \tilde{t} is identical, independent of the value of Ω_0 . We have shown above that this concept of “families” can be extended to include the much wider range of cosmological models considered here, parametrized by Ω_0 , Ω_{X0} , and n . For category 1 models, of which the matter-dominated are one example, there are also just a finite number of families per model (i.e. for each value of n) as in the matter-dominated case. For category 2 models, however, each value of κ corresponds to a different family. This might look like a mathematical trick at first sight, where we use equation (71) to group different models that appear to be unrelated. However, the fact that, within a given family, the models will be identical in supercomoving variables indicates that the various models assembled into the same family are indeed deeply related. For instance, for any particular value of n , the flat (zero-curvature) models constitute a family, defined by $\kappa = 0$. Hence, all flat models with the same value of n will have the same solution for $a(\tilde{t})$ in supercomoving variables. Other physically interesting families include critical models with a cosmological constant, defined by $n = 0$, $\kappa = -3/4^{1/3}$, and critical models with domain walls, defined by $n = 1$, $\kappa = -2$. Now that we have derived equation (71) and defined the concept of family, we are ready to solve the Friedmann equation for the various cases which have elementary solutions.

7.3. Solutions

7.3.1. Matter-Dominated Universe ($\Omega_{X0} = 0$)

These are the only models considered in S80. The solutions of the Friedmann equation are

$$a = \begin{cases} (\tilde{t}^2 - 1)^{-1}, & \Omega_0 < 1; \\ \tilde{t}^{-2}, & \Omega_0 = 1; \\ (\tilde{t}^2 + 1)^{-1}, & \Omega_0 > 1, \end{cases} \quad (72)$$

where the present value a_0 of the scale factor is

$$a_0 = \begin{cases} (1 - \Omega_0)/\Omega_0, & \Omega_0 < 1; \\ 1, & \Omega_0 = 1; \\ (\Omega_0 - 1)/\Omega_0, & \Omega_0 > 1. \end{cases} \quad (73)$$

These values of a_0 were chosen in order to eliminate the dependences upon Ω_0 in equation (72). These supercomoving-variable solutions are remarkably simple compared with their comoving-variable counterparts.

In equation (72), \tilde{t} is negative and increases with time from the value $\tilde{t} = -\infty$ at the Big Bang. For the flat ($\Omega_0 = 1$) model, $\tilde{t} = -1$ at the present, and approaches 0 as the universe expands to infinite radius. For the open ($\Omega_0 < 1$) model, $\tilde{t} < -1$ at the present, and approaches -1 as the universe expands to infinite radius. For the closed ($\Omega_0 > 1$) model, $\tilde{t} < 0$ at present, the universe reaches a maximum expansion at $\tilde{t} = 0$ and recollapses at $\tilde{t} = \infty$ (“the Big Crunch”). The solutions of equation (72) for the cosmic scale factor $a(\tilde{t})$ as a function of \tilde{t} are plotted in Figure 1a (solid curves). The solid dot on the curve for the flat model indicates the present. For all other models, the location of the present is a function of Ω_0 . In Figure 1b, we have plotted the “supermoving Hubble parameter” $\mathcal{H} = a^{-1}da/d\tilde{t}$ defined by equation (31), also using solid curves.

7.3.2. Universe with Infinite Strings ($n = 2$)

This case is extremely simple. Setting $n = 2$ in equation (61) has exactly the same effect as setting $\Omega_{X0} = 0$. Hence, the solutions for a universe with strings are exactly the same as equations (72) and (73) above for a matter-dominated universe, independent of the value of Ω_{X0} . This results from the fact that a uniform background term with $n = 2$ behaves exactly like the curvature term in the Friedmann equation. Hence, the solid curves plotted in Figure 1 for the matter-dominated models apply to these models as well.

7.3.3. Universe with a Nonclumping, Nonrelativistic Component ($n = 3$)

This case is also very simple. We define an effective density parameter Ω'_0 as

$$\Omega'_0 = \Omega_0 + \Omega_{X0}. \quad (74)$$

Equation (61) then reduces to

$$\tau = \frac{1}{2} \left(\frac{\Omega_0}{\Omega'_0} \right)^{1/2} \int dx \left(\frac{1 - \Omega'_0}{\Omega'_0} x^4 + x^3 \right)^{-1/2}. \quad (75)$$

This equation has exactly the same form as in the case of a matter-dominated universe, except for the extra factor $(\Omega_0/\Omega'_0)^{1/2}$ in front of the integral. However, this extra factor disappears after we eliminate τ using equations (30) and (60), and we get exactly the same solutions as for the matter dominated case,

$$a = \begin{cases} (\tilde{t}^2 - 1)^{-1}, & \Omega_0 + \Omega_{X0} < 1; \\ \tilde{t}^{-2}, & \Omega_0 + \Omega_{X0} = 1; \\ (\tilde{t}^2 + 1)^{-1}, & \Omega_0 + \Omega_{X0} > 1; \end{cases} \quad (76)$$

where the present value of the scale factor is given by

$$a_0 = \begin{cases} \frac{1 - \Omega_0 - \Omega_{X0}}{\Omega_0 + \Omega_{X0}}, & \Omega_0 + \Omega_{X0} < 1; \\ 1, & \Omega_0 + \Omega_{X0} = 1; \\ \frac{\Omega_0 + \Omega_{X0} - 1}{\Omega_0 + \Omega_{X0}}, & \Omega_0 + \Omega_{X0} > 1. \end{cases} \quad (77)$$

This was the reason for treating the case $n = 3$ differently in equations (25) and (60). Otherwise, the solutions would have shown dependences upon Ω_0 and Ω_{X0} . The solutions (76) and (77) are identical to

the solutions (72) and (73) for the matter-dominated cases (and hence, the solid curves plotted in Figure 1 apply to these models as well) if we replace Ω_0 in the latter by Ω'_0 . This makes sense since, for $n = 3$, the clumping, nonrelativistic and nonclumping, nonrelativistic X components are indistinguishable in the Friedmann equation. They are distinguishable in the Poisson equation, however, since one component can contribute to the peculiar gravitational potential while the other cannot. Hence, we end up with a different form for the Poisson equation, which is the price we must pay for having no dependence in equation (76) upon the cosmological parameters.

This modification of the Poisson equation is actually physically interesting. We can rewrite equation (39) as

$$|\tilde{\nabla}^2 \tilde{\phi}| = 6a \left| \frac{\tilde{\rho} + \tilde{\rho}_X}{\tilde{\rho} + \tilde{\rho}_X} - 1 \right| < 6a \left| \frac{\tilde{\rho}}{\tilde{\rho}} - 1 \right|. \quad (78)$$

Since the solution $a(\tilde{t})$ is the same for both matter dominated and $n = 3$ models, the interpretation of the inequality in equation (78) is that *the presence of the X component effectively weakens the peculiar gravity*. This explains, for example, why the growth of density fluctuations in the baryons and CDM is reduced in a universe which includes massive neutrinos, the Cold + Hot Dark Matter (CHDM) model, compared with the growth in a CDM model with the same Ω_0 , for wavelengths smaller than the neutrino free-streaming length. In the CHDM model, the growth of small-wavelength fluctuations occurs as if the neutrinos corresponded to a nonclumping, nonrelativistic background with $\Omega_\nu = \Omega_{X0}$, until very late times when long wavelength neutrino density fluctuations achieve significant growth too.

7.3.4. Universe with a Radiation Background ($n = 4$)

The solution of equation (61) for $n = 4$ is

$$a = e^{-2\tilde{t}} \left[\frac{(e^{-2\tilde{t}} - 1)^2}{4} - \frac{\Omega_{X0}(1 - \Omega_0 - \Omega_{X0})}{\Omega_0^2} \right]^{-1}, \quad (79)$$

where the present value a_0 of the scale factor is

$$a_0 = \frac{\Omega_0}{\Omega_{X0}}. \quad (80)$$

As in other cases, this solution gives $a = 0$ (Big Bang) for $\tilde{t} = -\infty$. The behavior of the solution depends upon the curvature of the universe. If $\Omega_0 + \Omega_{X0} < 1$, a becomes infinite when the square bracket in equation (79) is zero. This gives a maximum value for \tilde{t} ,

$$\tilde{t}(a = \infty) = -\frac{1}{2} \ln \left\{ 1 + \left[\frac{4\Omega_{X0}(1 - \Omega_0 - \Omega_{X0})}{\Omega_0^2} \right]^{1/2} \right\}. \quad (81)$$

If $\Omega_0 + \Omega_{X0} > 1$, the square bracket in equation (79) can never be 0, implying that a can never be infinite. This case corresponds to that of a bound universe. The Big Crunch ($a = 0$) occurs at $\tilde{t} = +\infty$. The time at turnaround can be found by setting $da/d\tilde{t} = 0$. We find

$$\tilde{t}_{\text{turnaround}} = -\frac{1}{4} \ln \left[1 + \frac{4\Omega_{X0}(\Omega_0 + \Omega_{X0} - 1)}{\Omega_0^2} \right]. \quad (82)$$

Unlike in the previous cases, we have not succeeded in obtaining a solution that has no dependences upon the parameters Ω_0 and Ω_{X0} . However, the solution (79) will be the same for different models with the same value of $\kappa = \Omega_{X0}(1 - \Omega_0 - \Omega_{X0})/\Omega_0^2$, which is precisely what equation (71) predicted. The essential difference between this case and the previous cases is that matter-dominated models, models with cosmic strings, and models with a uniform nonrelativistic component are one-parameter models (the parameter being Ω_0 for the first two cases, and $\Omega'_0 \equiv \Omega_0 + \Omega_{X0}$ for the latter case). It is then possible, as we saw, to find solutions with no explicit dependence upon that parameter. The radiation model, however, is a two-parameter model, and we cannot eliminate these parameters from the solution. The solution to this problem, as we saw in §7.2, is to consider models such as the radiation model, not as two-parameter models, but as an ensemble of one-parameter models, each model being characterized by the value of the constant κ in equation (71). Equation (79) then reduces to

$$a = e^{-2\tilde{t}} \left[\frac{(e^{-2\tilde{t}} - 1)^2}{4} - \kappa \right]^{-1}. \quad (83)$$

For any particular value of κ , we are in a family for which the cosmological parameters Ω_0 and Ω_{X0} are related by equation (71). All these models have the same solution, equation (83), independent of the value of the cosmological parameters. We have plotted the solution for flat models ($\kappa = 0$) in Figure 1 (dotted curves). The radiation component dominates at early times, which explains why this model has a different behavior at early time compared to all other models. The most interesting family is the one for which the spatial curvature is zero, which corresponds to $\kappa = 0$. In the limit $\tilde{t} \rightarrow 0^-$, we recover the solution $a = \tilde{t}^{-2}$, for the flat matter-dominated model, since the relative contribution of the radiation component decays as the universe expands.

7.3.5. Critical Universe with a Nonzero Cosmological Constant ($n = 0$, $\Omega_{X0} = \Omega_{X0,\text{crit}}$)

A universe with a positive cosmological constant is marginally bound if Ω_0 and Ω_{X0} satisfy equation (68) for $n = 0$. We substitute this expression into equation (61). The resulting integral has an elementary solution, which is

$$\tilde{t} = -\left(1 + \frac{1}{a}\right)^{1/2} - \frac{1}{3^{1/2}} \ln \frac{a + 1 - [3a(a + 1)]^{1/2}}{a + 1 + [3a(a + 1)]^{1/2}}, \quad (84)$$

where the present scale factor is given by

$$a_0 = \left(\frac{\Omega_{X0}}{4\Omega_0}\right)^{1/3}. \quad (85)$$

Unfortunately, we cannot invert equation (84) to express a as a function of \tilde{t} (except numerically). In this model, $\tilde{t} = -\infty$ at the Big Bang, and goes to $\tilde{t} = +\infty$ as the scale factor a approaches its maximum value $a_{\text{max}} = 1/2$. We have plotted this solution in Figure 1.

7.3.6. Critical Universe with Domain Walls ($n = 1$, $\Omega_{X0} = \Omega_{X0,\text{crit}}$)

A universe with domain walls is marginally bound if Ω_0 and Ω_{X0} satisfy equation (68) for $n = 1$. We substitute this expression in equation (61). The resulting integral has an elementary solution, which is

$$\tilde{t} = -\frac{1}{a^{1/2}} + \tanh^{-1} a^{1/2}, \quad (86)$$

where the present scale factor is given by

$$a_0 = \left(\frac{\Omega_{X0}}{\Omega_0} \right)^{1/2}. \quad (87)$$

Again, we cannot invert equation (86) to express a as a function of \tilde{t} . In this model, $\tilde{t} = -\infty$ at the Big Bang, and goes to $\tilde{t} = +\infty$ as the scale factor a approaches its maximum value $a_{\max} = 1$. We have plotted this solution in Figure 1

7.3.7. Flat Models with Domain Walls or a Cosmological Constant ($\Omega_0 + \Omega_{X0} = 1$, $n = 1$ or 0)

These cases are interesting and important for several reasons. First, the flatness of the cosmological model is usually regarded as a requirement of the standard inflationary scenario. Second, as summarized by Ostriker & Steinhardt (1995), the flat model with nonzero cosmological constant and Cold Dark Matter is of great interest as an explanation for several observations which are otherwise discordant with the standard flat CDM model without a cosmological constant. For example, the presence of a cosmological constant (or domain walls) increases the age of the universe for a given observed H_0 , helping to solve the so-called age problem. As a practical matter, therefore, it is convenient that, according to the discussion presented in §7.2, solutions of the Friedmann equation for flat models, elementary or not, will have no explicit dependence upon the cosmological parameters.

Equation (61) for flat models reduces to

$$\tau = \frac{1}{2} \int dx \left(x^3 + \frac{\Omega_{X0}}{\Omega_0} x^{6-n} \right)^{-1/2}. \quad (88)$$

Using equations (59) and (60), we can write this integral as

$$\tilde{t} = \frac{1}{2} \int_1^a \frac{dy}{y^{3/2}(1+y^{3-n})^{1/2}}. \quad (89)$$

where the present scale factor is defined by

$$a_0 = \left(\frac{\Omega_{X0}}{\Omega_0} \right)^{1/(3-n)}. \quad (90)$$

By writing equation (89) as a definite integral, we have fixed the value of the integration constant by imposing (arbitrarily) $a = 1$ at $\tilde{t} = 0$. As expected, equation (89) has no explicit dependence upon Ω_0 and Ω_{X0} . Unfortunately, this integral has no elementary solutions for the cases $n = 0$ and $n = 1$, and thus has to be evaluated numerically. In the limit $a \ll 1$, the term y^{3-n} can be neglected, and we recover the solution $\tilde{t} = -a^{1/2}$, $a = \tilde{t}^{-2}$ of the matter-dominated flat universe (eq. [72]). This was expected, since the contribution of the uniform component is negligible at early times for $n < 3$. The value of \tilde{t} at present is obtained by integrating equation (89) numerically for $a = a_0$. The results are given in Table 1 for various cases. Finally, \tilde{t} approaches a finite value as the universe expands to infinite radius. This value can be computed numerically by integrating equation (89) with $a = \infty$. The results are also given in Table 1.

7.3.8. Solution Families

The results presented in this section are summarized in Figure 2, where we plot the relationships between Ω_0 and Ω_{X0} which define family membership for each value of n , for models with $n = 0, 1, 2, 3$, and 4. For models with $n = 0, 1$, and 4, families are defined by equation (71), with each possible value of κ corresponding to a particular family. Families are thus represented by curves of constant κ . We have plotted several of these curves in Figure 2a, 2b, 2c, and 2f, and indicated the corresponding values of κ . Thick dashed curves indicate particular families that were discussed in this section (flat and critical models with a cosmological constant or domain walls, and flat models with radiation). The cases $n = 2$ and $n = 3$ are different. As we saw, there are only 3 families for these models. These families are represented in Figure 2d and 2e by the dashed line, and the half-planes on either one side of it. The dashed areas in Figure 2a and 2c represent regions in parameter space that are excluded by the existence of the Big Bang. In these regions, the Friedmann equation (4) predicts a minimum value $a = a_{\min} > 0$ for the scale factor. For $a < a_{\min}$, equation (4) gives $(\dot{a})^2 < 0$.

The solutions presented in this section describe the evolution of the cosmological background. The next step consists of describing the growth of fluctuations leading to the formation of large-scale structure. This problem is much too complex to be solved analytically for the general case. In the next two sections, we consider two particular cases of great cosmological importance for which analytical solutions exist, either because various terms in the fluid equations can be neglected, or because the problem has a high degree of symmetry. These two problems are: (1) the growth of fluctuations in the linear regime, and (2) the growth of a 1D, plane-wave density fluctuation from the linear to the nonlinear regime, leading to the formation of a collisionless pancake. The extension of these results to the nonlinear regime for a general distribution of initial density fluctuations is then made possible by the well-known Zel’dovich approximation presented in supercomoving variables in §10.

8. LINEAR PERTURBATION THEORY

We can use the fluid conservation equations in supercomoving variables to describe the growth of small-amplitude density fluctuations in the early universe. First, we decompose the density $\tilde{\rho}$ and the pressure \tilde{p} into a uniform and a space-varying component,

$$\tilde{\rho} = \tilde{\bar{\rho}}(1 + \delta), \quad (91)$$

$$\tilde{p} = \tilde{\bar{p}} + \tilde{\delta p}, \quad (92)$$

where $\tilde{\bar{\rho}}$ and $\tilde{\bar{p}}$ are the mean values of the density and pressure. In supercomoving variables, $\tilde{\bar{\rho}}$ is constant and $\tilde{\bar{p}}$ is a function of time only (see eq. [44]) (and $\tilde{\bar{p}} = \text{constant}$ if $\gamma = 5/3$). We substitute equations (91) and (92) into the fluid equations. These equations reduce to

$$\frac{\partial \delta}{\partial t} + \tilde{\nabla} \cdot [\tilde{\mathbf{v}}(1 + \delta)] = 0, \quad (93)$$

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}} = -\frac{\tilde{\nabla} \tilde{\delta p}}{\tilde{\bar{\rho}}(1 + \delta)} - \tilde{\nabla} \tilde{\phi}, \quad (94)$$

$$\tilde{\nabla}^2 \tilde{\phi} = 6a f_n \delta. \quad (95)$$

A trivial case which satisfies these equations is $\delta = 0$, $\tilde{\mathbf{v}} = 0$, $\tilde{\phi} = 0$, $\tilde{\delta p} = 0$, which corresponds to a universe with no fluctuation. Assuming that the fluctuations are small, we can solve equations (93)–(95), in general,

using perturbation theory. After dropping second order terms, equations (93) and (94) reduce to

$$\frac{\partial \delta}{\partial t} + \tilde{\nabla} \cdot \tilde{\mathbf{v}} = 0, \quad (96)$$

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} = -\frac{\tilde{\nabla} \tilde{\delta p}}{\tilde{\rho}} - \tilde{\nabla} \tilde{\phi}. \quad (97)$$

(Eq. [95] is already linear.) To solve these equations, we take the divergence of equation (97), then use equation (96) to eliminate $\tilde{\mathbf{v}}$, and equation (95) to eliminate $\tilde{\phi}$. We get

$$\frac{\partial^2 \delta}{\partial t^2} = \frac{\tilde{\nabla}^2 \tilde{\delta p}}{\tilde{\rho}} + 6a f_n \delta. \quad (98)$$

To solve this equation, we need a relationship between the pressure fluctuation $\tilde{\delta p}$ and density contrast δ . For adiabatic perturbations, $\tilde{\nabla}^2 \tilde{\delta p} = \tilde{c}_s^2 \tilde{\rho} \tilde{\nabla}^2 \delta$ to first order, where $\tilde{c}_s = (\gamma \tilde{p} / \tilde{\rho})^{-1/2}$ is the mean sound speed. Equation (98) reduces to

$$\frac{\partial^2 \delta}{\partial t^2} = \tilde{c}_s^2 \tilde{\nabla}^2 \delta + 6a f_n \delta. \quad (99)$$

8.1. Critical Jeans Length

The density contrast δ can be expressed as a sum of plane waves,

$$\delta = \sum_{\mathbf{k}} \delta_{\mathbf{k}} e^{i\mathbf{k} \cdot \tilde{\mathbf{r}}}. \quad (100)$$

We substitute equation (100) in equation (99). The terms for each value of \mathbf{k} must cancel out separately. We then get a separate equation for each mode \mathbf{k} ,

$$\frac{\partial^2 \delta_{\mathbf{k}}}{\partial t^2} = 6a f_n \left[1 - \left(\frac{\tilde{\lambda}_J}{\tilde{\lambda}} \right)^2 \right] \delta_{\mathbf{k}}, \quad (101)$$

where $\tilde{\lambda} = 2\pi/|\mathbf{k}|$ is the wavelength of the mode, and $\tilde{\lambda}_J$ is the Jeans wavelength, defined by

$$\tilde{\lambda}_J = 2\pi(6a f_n)^{-1/2} \tilde{c}_s. \quad (102)$$

Perturbations can grow only if $\tilde{\lambda} > \tilde{\lambda}_J$. Modes with $\tilde{\lambda} < \tilde{\lambda}_J$ result in sound waves. The mean sound speed is constant for $\gamma = 5/3$ models, and a function of a alone for $\gamma \neq 5/3$ models. Thus, the only time dependence in equation (102) is contained in the scale factor $a(\tilde{t})$. This implies that within any given family, the value of the Jeans length is independent of the cosmological parameters. For instance, we can compute the Jeans length for matter-dominated models by substituting equation (72) into equation (102). We get

$$\tilde{\lambda}_J = \frac{2\pi}{6^{1/2}} \tilde{c}_{s,0} a_0^{(3\gamma-5)/2} \times \begin{cases} (\tilde{t}^2 - 1)^{3\gamma/2-2}, & \Omega_0 < 1; \\ \tilde{t}^{3\gamma-4}, & \Omega_0 = 1; \\ (\tilde{t}^2 + 1)^{3\gamma/2-2}, & \Omega_0 > 1. \end{cases} \quad (103)$$

We note that, if the temperature of the baryon-electron gas remains coupled to that of the cosmic microwave radiation background by Compton scattering, then $\gamma = 4/3$ is the appropriate value to use in

equation (43) and $\tilde{c}_s \propto a^{1/2}$. In that case, equation (102) indicates that $\tilde{\lambda}_J$ is independent of time. This means that the baryon Jeans *mass* M_J (in physical units, where $M_J \propto \bar{\rho}\lambda_J^3$) must be constant in time, as well, in that case. This latter result is a well-known one for the Einstein-de Sitter model, and is relevant to the postrecombination epoch between $z \sim 10^3$ and $z \sim 10^2$ for mean baryon densities consistent with Big Bang nucleosynthesis constraints for the observed values of the Hubble constant (see, for example, Shapiro, Giroux, & Babul 1994). However, by solving the problem in supercomoving variables, we have generalized this result to a very much wider class of models.

8.2. Zero Pressure Solutions

In the limit $\lambda \gg \lambda_J$, equation (99) reduces to

$$\frac{\partial^2 \delta}{\partial \tilde{t}^2} = 6a f_n \delta. \quad (104)$$

This equation has a slightly simpler form in supercomoving variables than in comoving variables. However, both equations are equally complicated to solve. Indeed, the first step in solving equation (104) is to replace the independent variable \tilde{t} by $x = a/a_0$ using equations (25), (32), and (33). Equation (104) reduces to

$$\begin{aligned} [(1 - \Omega_0 - \Omega_{X0})x^3 &+ \Omega_0 x^2 + \Omega_{X0} x^{5-n}] \delta'' \\ &+ \left[2(1 - \Omega_0 - \Omega_{X0})x^2 + \frac{3}{2}\Omega_0 x + \left(3 - \frac{n}{2}\right)\Omega_{X0} x^{4-n} \right] \delta' = \frac{3\Omega_0}{2} \delta, \end{aligned} \quad (105)$$

where a prime stands for d/dx (notice that f_n has disappeared from the equation). This equation is the same in supercomoving and comoving variables. It has two independent solutions, a decaying solution δ_- and a growing solution δ_+ . We shall focus on the growing solution, which is responsible for the formation of structures. Numerous solutions of equation (105) have been published. Hence, we only need to reexpress these known solutions in terms of supercomoving variables, by eliminating the scale factor $a(\tilde{t})$ using the solutions derived in §7.3.

8.2.1. Matter-Dominated Universe ($\Omega_{X0} = 0$)

The solutions for matter-dominated models are given in Weinberg (1972) and Groth & Peebles (1975). In terms of supercomoving variables these solutions are

$$\delta_+ = \begin{cases} \frac{5}{2} \left[1 + 3(\tilde{t}^2 - 1) \left(1 - \tilde{t} \tanh^{-1} \frac{1}{\tilde{t}} \right) \right], & \Omega_0 < 1; \\ \tilde{t}^{-2}, & \Omega_0 = 1; \\ \frac{5}{2} \left[-1 + 3(\tilde{t}^2 + 1) \left(1 - \tilde{t} \tanh^{-1} \frac{1}{\tilde{t}} \right) \right], & \Omega_0 > 1; \end{cases} \quad (106)$$

where the factors of 5/2 were chosen such that all solutions reduce to $\delta_+ = \tilde{t}^{-2}$ at early time. These solutions were given in S80 (Actually, S80 eq. [15] contains some typographical errors. We have corrected this in our eq. [106] above).

8.2.2. Universe with Infinite Strings ($n = 2$)

As in the case of the Friedmann equation, setting $n = 2$ in equation (105) has the same effect as setting $\Omega_{X0} = 0$, hence the solutions (106) also apply to models with infinite strings.

8.2.3. Universe with a Nonzero Cosmological Constant ($n = 0$)

The general solutions of the perturbation equation for models with a nonzero cosmological constant are given in Edwards & Heath (1976), P80, Martel (1991), Lahav et al (1991), and Bildhauer, Buchert, & Kasai (1992) (see also Carroll, Press, & Turner 1992) For flat models, the solution is

$$\delta_+ = \frac{5}{2} \left(1 + \frac{1}{a^3} \right)^{1/2} \int_0^a \frac{w^{3/2} dw}{(1 + w^3)^{3/2}}, \quad (107)$$

and for critical models,

$$\delta_+ = 5 \left(\frac{1 - 3a + 4a^3}{4a^3} \right)^{1/2} \int_0^a \frac{w^{3/2} dw}{(1 + w)^{3/2} (1 - 2w)^3}. \quad (108)$$

Since a is not an elementary function of \tilde{t} for these models, we cannot express δ_+ as a function of \tilde{t} in closed form. Notice, however, that supercomoving variables still played an important role. If we had not been using the definitions of a_0 given by equations (85) and (90), the solutions (107) and (108) would show explicit dependences upon Ω_0 and Ω_{X0} , as they normally do in comoving variables.

8.2.4. Universe with a Radiation Background ($n = 4$)

Solutions for models with a radiation background were derived by Mészáros (1974) and Groth & Peebles (1975). In supercomoving variables, the growing solution for flat models (that is, $\kappa = 0$ in eq. [83]) is

$$\delta_+ = 1 + \frac{6e^{-2\tilde{t}}}{(e^{-2\tilde{t}} - 1)^2}. \quad (109)$$

8.2.5. Universe with a Nonclumping, Nonrelativistic Component ($n = 3$)

Models with a nonclumping, nonrelativistic component ($n = 3$) were discussed by Bond, Efstathiou, & Silk (1980) and Wasserman (1981). Such models are of interest in the context of models like the Cold+Hot Dark Matter model (CHDM) in which one component of matter, the Hot Dark Matter, has a much larger effective Jeans length than the others. In that case, while fluctuations are able to grow in the other components on scales smaller than this effective Jeans length in the HDM, they do not grow in the HDM. The $n = 3$ model, therefore, becomes a useful approximation for small wavelengths in the CHDM model. For flat models, the solution is

$$\delta_+ = a^m = (-\tilde{t})^{-m/2}, \quad (110)$$

where

$$m = \frac{1}{4} \left[-1 + (1 + 24\Omega_0)^{1/2} \right]. \quad (111)$$

For non-flat models, the solutions are not power laws. In the limit $a/a_0 \ll 1$, however, these solutions approach a power law, but with a different exponent,

$$m = \frac{1}{4} \left[-1 + \left(1 + \frac{24\Omega_0}{\Omega_0 + \Omega_{X0}} \right)^{1/2} \right], \quad a \ll a_0. \quad (112)$$

Notice that the solutions (110)–(112) do not approach $\delta_+ \propto a$ at early time as they do in all models with $n < 3$. Hence, there is no preferred normalization for these solutions.

Unlike in the previous cases, the solutions (110)–(112) explicitly depend upon the density parameter Ω_0 . In all the previous cases, the solutions were not power laws. Hence, there was a natural time scale in each solution, enabling us to rescale the time variable in order to make these solutions identical. But in the case of equation (110), the solution is a power law, and therefore has no natural time scale that could be used for rescaling. Hence, the solutions (110) for different values of Ω_0 are genuinely different, and cannot be transformed into one another by a rescaling of the variables.

8.2.6. Universe with Domain Walls ($n = 1$)

For models with domain walls ($n = 1$), we could not find any published solution in the literature. We shall therefore derive our own solutions of equation (105) for $n = 1$. As in §7.3, we consider two specific cases, flat models and critical models.

First, consider as a trial solution the decaying solution for the Einstein-de Sitter model ($\Omega_0 = 1$, $\Omega_{X0} = 0$), that is, $\delta_- = x^{-3/2}$. We substitute this solution into equation (105), and get

$$\frac{3}{4}(1 - \Omega_0 - \Omega_{X0})x^{1/2} + \frac{3}{4}\Omega_{X0}(n - 1)x^{5/2-n} = 0. \quad (113)$$

This trial solution is the correct solution if both terms in equation (113) vanish. The first term vanishes for flat models. The second term vanishes either for $\Omega_{X0} = 0$ or $n = 1$. The former case corresponds to the Einstein-de Sitter model, the latter to a model with domain walls. Hence *the decaying mode is the same in a flat universe with domain walls as in a flat universe with no X component*. Of course, this is true only if we express δ_- as a function of a (or $x \equiv a/a_0$). The relationship between a and \tilde{t} is model-dependent, and so is the relationship between δ and \tilde{t} .

We thus have found the decaying mode for flat models with domain walls. To find the growing mode, we set $\delta = \delta_- \int f(x)dx$ and substitute in equation (105), with $1 - \Omega_0 - \Omega_{X0} = 0$ and $n = 1$. We get a first order equation for $f(x)$, which is easily solved. The final solution is

$$\delta_+ = \frac{5}{2}a^{-3/2} \int_0^a \frac{w^{3/2}dw}{(1 + w^2)^{1/2}}. \quad (114)$$

For critical models with domain walls, we could not find any elementary solution for equation (106). However, we can express the growing solution as an infinite series. Since the solution must take the form $\delta_+ \propto a$ in the limit $a \ll 1$ (this is true for all models with $n < 3$), we define

$$\delta_+ = \sum_{l=0}^{\infty} c_l a^{l+1}. \quad (115)$$

We substitute this solution into the perturbation equation, and solve for the coefficients. The solution is

$$c_0 = 1, \quad c_1 = \frac{8}{7}, \quad c_l = \frac{4l(l+1)c_{l-1} + (-2l^2 + l + 1)c_{l-2}}{l(2l+5)}, \quad l \geq 2. \quad (116)$$

As in all the previous cases with $n < 3$, we normalize these solutions by imposing $\delta_+ = \tilde{t}^{-2}$ at early time.

9. THE COLLISIONLESS PANCAKE PROBLEM

For a 1D, planar density perturbation, it is possible to solve the fully nonlinear problem exactly, up to the moment of caustic formation, in the limit in which pressure can be neglected (Sunyaev & Zel’dovich 1972; Doroshkevich, Ryabenkii, & Shandarin 1973). This is the problem which is sometimes described as the cosmological “pancake” problem (Zel’dovich 1970). Writing this solution in supercomoving variables serves to demonstrate the universal character of the solution, independent of cosmological model, as follows. We consider, for simplicity, a 1D, primordial plane-wave density fluctuation as the initial condition, which is a solution of the linear perturbation equations discussed in the previous section.

In the collisionless limit ($\tilde{p} = 0$), the fluid conservation equation in supercomoving form, for plane-symmetric systems, reduce to

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}(\tilde{\rho} \tilde{v}) = 0, \quad (117)$$

$$\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{x}} = -\frac{\partial \tilde{\phi}}{\partial \tilde{x}}, \quad (118)$$

$$\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} = 6af_n \left(\frac{\tilde{\rho}}{\bar{\rho}} - 1 \right). \quad (119)$$

We now set up initial conditions at time \tilde{t}_i by displacing each mass element from its unperturbed location \tilde{q} by a small amount $\tilde{\xi}_i$, as follows,

$$\tilde{x}_i = \tilde{q} - \tilde{\xi}_i. \quad (120)$$

For the particular problem of the collisionless pancake, we choose initial conditions that are periodic,

$$\tilde{\xi}_i = \frac{b_i}{2\pi\tilde{k}} \sin 2\pi\tilde{k}\tilde{q}, \quad (121)$$

where \tilde{k} is the wavenumber of the perturbation. The corresponding initial density profile is given by

$$\tilde{\rho}_i = \frac{\bar{\rho}}{1 - b_i \cos 2\pi\tilde{k}\tilde{q}} \simeq \bar{\rho}(1 + b_i \cos 2\pi\tilde{k}\tilde{q}), \quad (122)$$

where the last equality is valid in the limit $b_i \ll 1$. The initial density contrast is therefore given by $\delta_i = b_i \cos 2\pi\tilde{k}\tilde{q}$. Since the density contrast at early times grows according to the solution of the linear perturbation equations, the solution at early times must reduce to

$$\delta(\tilde{t}) = b_i \frac{\delta_+(\tilde{t})}{\delta_+(\tilde{t}_i)} \cos 2\pi\tilde{k}\tilde{q}. \quad (123)$$

Our goal is to find a solution of the fluid equations (117)–(119) which reduces to equation (122) at early times. The solution is given by

$$\tilde{\xi}(\tilde{x}, \tilde{t}) = \frac{b(\tilde{t})}{2\pi\tilde{k}} \sin 2\pi\tilde{k}\tilde{q}, \quad (124)$$

where the amplitude $b(\tilde{t})$ is equal to b_i at the initial time. We differentiate equation (120), and get

$$\tilde{v} = -\frac{db/d\tilde{t}}{2\pi\tilde{k}} \sin 2\pi\tilde{k}\tilde{q}, \quad (125)$$

We now substitute equation (125) into the continuity equation (117), and solve for the density, according to

$$\tilde{\rho} = \frac{\tilde{\rho}}{1 - b \cos 2\pi\tilde{k}\tilde{q}}. \quad (126)$$

We then substitute equations (125) and (126) into equations (118) and (119). We are left with two unknowns, $\tilde{\phi}$ and $b(\tilde{t})$. We eliminate $\tilde{\phi}$, and get

$$\frac{d^2b}{d\tilde{t}^2} = 6f_nab. \quad (127)$$

This equation is identical to equation (104). The solution is therefore given by

$$b(\tilde{t}) = A\delta_+(\tilde{t}) + B\delta_-(\tilde{t}). \quad (128)$$

By imposing the boundary condition (123), we get

$$A = b_i/\delta_+(\tilde{t}_i), \quad (129)$$

$$B = 0. \quad (130)$$

In supercomoving variables, the solutions for δ_+ do not depend upon the cosmological parameters within each family, except for the case $n = 3$. The solution (124)–(126) describes the growth of a sinusoidal perturbation that forms a “pancake,” that is, a caustic surface of infinite density, when the amplitude $b(\tilde{t})$ reaches unity. The solution is independent of the cosmological parameters within each family. When $b(\tilde{t})$ reaches unity, this solution breaks down. At this point, strong shock waves form on both sides of the pancake, and the collisionless approximation ($\tilde{\rho} = 0$) is not valid anymore. Notice that the fluid equations are exactly the same in supercomoving variables as in noncosmological variables. This implies that the jump conditions in supercomoving variables do not depend upon the cosmological parameters. Since the solutions before shock formation do not depend upon these parameters either, we conclude that the postshock solution will also be independent of the cosmological parameters within each family. This means, in particular, that the pancake collapse problem, in the absence of non-adiabatic physical processes like radiative cooling, has one solution only, when expressed in supercomoving variables. Hence, for pancakes occurring in different cosmological models, the values and spatial variations of any fluid variable at any time are exactly the same as long as length is expressed in units of the pancake wavelength while time is expressed in terms of the cosmic scale factor a/a_c , where a_c is the scale factor at which $b(\tilde{t}) = 1$ and caustics are predicted to occur as described above, independent of the cosmological model. Approximate analytical solutions of the pancake collapse problem both before and after shock formation will be presented in a forthcoming paper (Shapiro, Struck-Marcell, & Martel 1998).

10. THE ZEL'DOVICH APPROXIMATION

Zel'dovich (1970) presented an analytical approximation for the growth of density perturbations, which extrapolates the linear solution into the nonlinear regime. This approximation is exact in the linear regime and, for locally 1D motion, remains valid into the nonlinear regime, up to the formation of the first caustics. In this section, we derive the supercomoving form of the Zel'dovich approximation.

Let $\mathbf{r}(t)$ be the proper position of a mass element at proper time t , and let $\mathbf{r}_i = \mathbf{r}(t_i)$ be the position of that mass element at some initial time t_i , in the absence of perturbation. It can be shown that, at early time, the proper position \mathbf{r} is given by

$$\mathbf{r}(t) = a(t) \left[\mathbf{q} + \delta_+(\tilde{t}) \mathbf{p}(\mathbf{q}) \right], \quad (131)$$

where $\mathbf{q} \equiv \mathbf{r}_i/a(t_i)$, and \mathbf{p} is a vector field that is function of \mathbf{q} only (not t), and characterizes the perturbation. The quantity $\partial p_j / \partial q_k$ is sometimes referred to as the deformation tensor D_{jk} . The time-independence of \mathbf{p} implies that at early times mass elements are moving on straight lines. The Zel'dovich “approximation” consists of assuming that equation (131) is valid, not only at early times, but up to the point where mass element trajectories cross, at which point the density diverges. Writing equation (131) in supercomoving form yields

$$\tilde{\mathbf{r}}(\tilde{t}) = \tilde{\mathbf{q}} + \delta_+(\tilde{t}) \tilde{\mathbf{p}}(\tilde{\mathbf{q}}). \quad (132)$$

Notice that $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{p}}$ were computed using equation (15) *without* the factor of a , since \mathbf{q} and \mathbf{p} in equation (131) are already expressed in comoving variables. In supercomoving variables, $\tilde{\mathbf{q}} = [a/a(\tilde{t}_i)] \mathbf{r}_i$, and the deformation tensor is unchanged, $\tilde{D}_{jk} = D_{jk}$. As in the case of the collisionless pancake problem, we get a solution whose dependence upon the cosmological model is entirely contained in the growing mode δ_+ . Therefore, the solutions are identical within any given family.

11. SUMMARY AND CONCLUSION

In this paper, we described a set of dimensionless variables, which we call *supercomoving variables*, for the description of the state and evolution of matter in a cosmologically expanding Friedmann universe in the Newtonian approximation (i.e. nonrelativistic gas on subhorizon scales). These supercomoving variables have the following properties: (1) Thermodynamic variables which depend in general upon space and time are stationary in the absence of perturbation, so that the effects of universal adiabatic cosmic expansion are accounted for implicitly and do not appear explicitly as sources of time-dependence, as long as the ratio of specific heats is $\gamma = 5/3$. (2) Velocities reflect departures from uniform Hubble flow only and in the absence of peculiar gravity or pressure gradients, are stationary. (3) One advantage of these variables over standard comoving variables is that, for ideal gas with ratio of specific heats, $\gamma = 5/3$, the fluid conservation equations in supercomoving variables are identical to the Newtonian fluid conservation equations for a noncosmological gas, while in standard comoving variables, extra terms appear to describe the effects of cosmology. (4) Similarly, the collisionless Boltzmann equation in supercomoving variables is identical to that for a nonrelativistic, noncosmological, collisionless gas. (5) The choice of nondimensionalizing units removes explicit dependence of the variables and the equations which describe their evolution including the Poisson equation for peculiar gravitational field, on the Hubble constant H_0 .

Our main objective in deriving supercomoving variables was to obtain a set of cosmological fluid equations that resemble as closely as possible the fluid equations in noncosmological variables, and do not show any explicit dependences upon cosmological parameters such as H_0 , Ω_0 , Ω_{X0} , or n . This was first achieved for matter-dominated models by Shandarin (S80) for adiabatic gas, with terms for nonadiabatic physical

processes added by Shapiro et al. (1983) and Shapiro & Struck-Marcell (1985). Here we have extended this approach to include models with a generalized, smooth background energy density (such as a cosmological constant) as well as additional physical processes, like viscosity and magnetic fields. The continuity equation, the momentum equation, and the equation of state have exactly the same form in noncosmological and supercomoving variables. The cosmological drag term, that is present in the momentum equation in comoving variables, is absent in supercomoving variables. The energy equation has the same form in noncosmological and supercomoving variables, except for an additional drag term (which is also present in comoving variables). However, unlike the case of standard comoving variables, that drag term vanishes for the physically interesting case $\gamma = 5/3$. Only the Poisson equation remains very different from its noncosmological form. In supercomoving variables, as in comoving variables, the source of the gravitational potential is not the density, but rather the density contrast.

The supercomoving variables reveal the existence of similarities among various cosmological models. This was shown in S80 for matter-dominated models. In comoving variables, the fluid equations, and their solutions, depend explicitly upon the density parameter Ω_0 . In supercomoving variables, all matter-dominated models fall into three categories, called *families*, defined by $\Omega_0 < 1$, $\Omega_0 = 1$, and $\Omega_0 > 1$. Within each family, the fluid equations and their solutions are independent of Ω_0 . Therefore, using supercomoving variables effectively reduces the number of matter-dominated models from infinity to 3. The existence of these families has direct practical significance for the solution of problems involving gas dynamics and gravitational clustering in cosmology, as follows. Any solution of the differential equations in supercomoving variables obtained for a given initial or boundary value problem for a particular set of values of the cosmological parameters H_0 and Ω_0 for which those initial and boundary values have also been expressed in supercomoving variables immediately yields the solution for any other values of H_0 and Ω_0 , within the same family as well, as long as the supercomoving initial and boundary values are also the same. This “Family” membership, then implies a “similarity” solution for initial and boundary value problems, where the supercomoving variables are the similarity variables.

In this paper, we generalized supercomoving variables to cosmological models with two components, the usual nonrelativistic component, and a uniform component, called the X component, whose energy density varies as a^{-n} . This generic two-component model includes as special cases several physically motivated cosmological models, such as models with a nonzero cosmological constant, domain walls, cosmic strings, massive neutrinos, or a radiation background. Each of these model is a two-parameter model, defined by the density parameters Ω_0 and Ω_{X0} of the nonrelativistic and X components, respectively.

We extended the concept of family introduced in S80 for matter-dominated models. We showed the existence of similarities among models with different values of Ω_0 and Ω_{X0} . For any particular value of n , the various families are defined by the value of the parameter κ appearing in equation (71). Hence, each combination (n, κ) corresponds to a particular family, and, as we showed, the fluid equations and their solutions do not depend upon the cosmological parameters Ω_0 and Ω_{X0} within a particular family. While Shandarin (S80) reduced a one-parameter model (the parameter being Ω_0) to three no-parameter models, we are reducing two-parameter models (one for each value of n , the parameters being Ω_0 and Ω_{X0}) to one-parameter models (the parameter being κ). Therefore, as in the case of matter-dominated models, solutions obtained in supercomoving variables for a particular model (that is, for particular values of Ω_0 and Ω_{X0}) can be applied to other models as long as these models are in the same family. In this paper, we focused our attention on 14 particular families which are physically interesting: (1) the Einstein-de Sitter model, (2) open, matter-dominated models, (3) closed, matter-dominated models, (4) flat models with a nonclumping background of nonrelativistic matter, such as would describe the case of massive neutrinos in the small wavelength limit, (5) open models with a nonrelativistic matter background, such as massive neutrinos, (6) closed models with a nonrelativistic matter background, such as massive neutrinos, (7) marginally bound

models with cosmic strings, ² (8) unbound models with cosmic strings, (9) bound models with cosmic strings, (10) flat models with a radiation background, (11) flat models with a nonzero cosmological constant, (12) critical models (i.e. asymptotically static) with a nonzero cosmological constant, (13) flat models with domain walls, and (14) critical models (i.e. asymptotically static) with domain walls. Table 4 summarizes the values of the parameters that define these various families.

To illustrate the use of supercomoving variables and the value of the “family” membership they reveal, we have, for each of these families, described the linear growth of density fluctuations in supercomoving variables, the supercomoving Jeans length, and the nonlinear growth of 1D planar density fluctuations – the cosmological pancake problem – prior to caustic formation. The solutions of these problems are independent of the particular values of the cosmological parameters Ω_0 and Ω_{X0} within a given family, with the exception of the models with a nonrelativistic matter background such as massive neutrinos, for which the solutions must remain explicitly parametrized by Ω_0 and Ω_{X0} . The model with massive neutrinos differs from the other models in that the energy density of the two components both vary as a^{-3} . As a result, there is no characteristic time scale when the universe goes from being matter-dominated to being X component-dominated, or vice-versa. This absence of characteristic time scale implies that solutions for different models are genuinely different, and cannot be turned into one another by a scaling transformation, as for the other models. The only exception is the evolution of the scale factor (eq. [76]), which is made independent of Ω_0 and Ω_{X0} by our introduction of the parameter f_n in the definition of the fiducial time t_* (eq. [25]). By this choice, however, the form of the Poisson equation is different for $n = 3$ models.

Finally, we have shown how the Navier-Stokes equation and the viscous energy equation are expressed in supercomoving variables, and presented supercomoving forms for the inviscid equation for vorticity evolution and the MHD equations, including the Biermann battery term. The latter two make clear how the vorticity and cosmic magnetic fields generated by large-scale structure formation, as discussed recently by Kulsrud et al. (1997), are related to each other and how they scale with characteristic time and length scales for cosmic structure formation in most cosmological models of interest.

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²There is an important distinction, often forgotten, between the terms “closed” and “bound.” The former refers to the curvature of the universe, and is determined by the sign of $\Omega_0 + \Omega_{X0} - 1$; the latter refers to whether the universe will expand forever or eventually recollapse. For matter dominated models (and also models with massive neutrinos), closed models are always bound and open models are always unbound. But this is not true in general for other models.

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A. ADDITIONAL PHYSICAL PROCESSES

A.1. Heating, Cooling, and Thermal Conduction

Shapiro et al. (1983), and Shapiro & Struck-Marcell (1985) showed how to transform the terms for the additional physical processes of radiative cooling, external heating, and thermal conduction into supercomoving variables. This requires only a modification of the energy equation. In noncosmological variables, this equation becomes

$$\frac{\partial \varepsilon}{\partial t} + \mathbf{u} \cdot \nabla \varepsilon = -\frac{p}{\rho} \nabla \cdot \mathbf{u} + \frac{\Gamma - \Lambda}{\rho} - \frac{\nabla \cdot \mathbf{S}}{\rho}, \quad (\text{A1})$$

where Γ is the volume heating rate, Λ is the volume cooling rate, and \mathbf{S} is the conductive flux. We define the supercomoving heating rate $\tilde{\Gamma}$, cooling rate $\tilde{\Lambda}$, and conductive flux $\tilde{\mathbf{S}}$ as

$$\tilde{\Gamma} = \frac{a^7 \Gamma}{\Gamma_*}, \quad (\text{A2})$$

$$\tilde{\Lambda} = \frac{a^7 \Lambda}{\Lambda_*}, \quad (\text{A3})$$

$$\tilde{\mathbf{S}} = \frac{a^6 \mathbf{S}}{S_*}, \quad (\text{A4})$$

where

$$\Gamma_* = \Lambda_* \equiv \frac{p_*}{t_*}, \quad (\text{A5})$$

$$S_* \equiv \rho_* v_*^3. \quad (\text{A6})$$

Equation (A1) becomes, in supercomoving variables,

$$\frac{\partial \tilde{\varepsilon}}{\partial \tilde{t}} + \tilde{\mathbf{v}} \cdot \tilde{\nabla} \tilde{\varepsilon} = -\mathcal{H}(3\gamma - 5)\tilde{\varepsilon} - \frac{\tilde{p}}{\tilde{\rho}} \tilde{\nabla} \cdot \tilde{\mathbf{v}} + \frac{\tilde{\Gamma} - \tilde{\Lambda}}{\tilde{\rho}} - \frac{\tilde{\nabla} \cdot \tilde{\mathbf{S}}}{\tilde{\rho}}. \quad (\text{A7})$$

This equation has the same form as equation (A1), except for the drag term, which vanishes for $\gamma = 5/3$.

A.2. Viscosity

If the fluid has a finite viscosity, the momentum equation is replaced by the Navier-Stokes equation,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \Phi - \frac{\nabla p}{\rho} + \frac{\eta}{\rho} \nabla^2 \mathbf{u} + \frac{\zeta + \eta/3}{\rho} \nabla(\nabla \cdot \mathbf{u}), \quad (\text{A8})$$

(Landau & Lifshitz 1987) where η and ζ are the coefficients of viscosity. The presence of viscosity dissipates kinetic energy into heat, hence the energy equation must be modified accordingly. Its final form is

$$\frac{\partial \varepsilon}{\partial t} + \mathbf{u} \cdot \nabla \varepsilon = -\frac{p}{\rho} \nabla \cdot \mathbf{u} + \frac{1}{\rho} \sigma'_{ij} \frac{\partial u_i}{\partial r_j}, \quad (\text{A9})$$

where the stress tensor σ' is defined by

$$\sigma'_{ij} = \eta \left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial r_k} \right) + \zeta \delta_{ij} \frac{\partial u_k}{\partial r_k}. \quad (\text{A10})$$

We define the following supercomoving coefficients of viscosity and stress tensor:

$$\tilde{\eta} = \frac{a^3 \eta}{\eta_*}, \quad (\text{A11})$$

$$\tilde{\zeta} = \frac{a^3 \zeta}{\zeta_*}, \quad (\text{A12})$$

$$\tilde{\sigma}'_{ij} = \tilde{\eta} \left(\frac{\partial \tilde{v}_i}{\partial \tilde{r}_j} + \frac{\partial \tilde{v}_j}{\partial \tilde{r}_i} - \frac{2}{3} \delta_{ij} \frac{\partial \tilde{v}_k}{\partial \tilde{r}_k} \right) + \tilde{\zeta} \delta_{ij} \left(\frac{\partial \tilde{v}_k}{\partial \tilde{r}_k} + 6\mathcal{H} \right), \quad (\text{A13})$$

where

$$\eta_* = \zeta_* \equiv \frac{\rho_* r_*^2}{t_*}. \quad (\text{A14})$$

With these definitions, equations (A8) and (A9) become, in supercomoving variables,

$$\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + (\tilde{\mathbf{v}} \cdot \tilde{\nabla}) \tilde{\mathbf{v}} = -\tilde{\nabla} \tilde{\phi} - \frac{\tilde{\nabla} \tilde{p}}{\tilde{\rho}} + \frac{\tilde{\eta}}{\tilde{\rho}} \tilde{\nabla}^2 \tilde{\mathbf{v}} + \frac{\tilde{\zeta} + \tilde{\eta}/3}{\tilde{\rho}} \tilde{\nabla} (\tilde{\nabla} \cdot \tilde{\mathbf{v}}), \quad (\text{A15})$$

$$\frac{\partial \tilde{\varepsilon}}{\partial \tilde{t}} + \tilde{\mathbf{v}} \cdot \tilde{\nabla} \tilde{\varepsilon} = -\mathcal{H}(3\gamma - 5) \tilde{\varepsilon} - \frac{\tilde{p}}{\tilde{\rho}} \tilde{\nabla} \cdot \tilde{\mathbf{v}} + \frac{1}{\tilde{\rho}} \tilde{\sigma}'_{ij} \frac{\partial \tilde{\mathbf{v}}_i}{\partial \tilde{x}_j} + \frac{9\tilde{\zeta}}{\tilde{\rho}} \mathcal{H}^2. \quad (\text{A16})$$

A.3. Vorticity

The vorticity is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}. \quad (\text{A17})$$

In the absence of viscosity, the vorticity evolves according to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = -\nabla p \times \nabla \left(\frac{1}{\rho} \right). \quad (\text{A18})$$

To convert these equations to supercomoving form, we define

$$\tilde{\boldsymbol{\omega}} = \frac{a^2 \boldsymbol{\omega}}{\omega_*}, \quad (\text{A19})$$

where

$$\omega_* \equiv \frac{1}{t_*}. \quad (\text{A20})$$

Equations (A17) and (A18) become, in supercomoving variables,

$$\tilde{\boldsymbol{\omega}} = \tilde{\nabla} \times \tilde{\mathbf{v}}, \quad (\text{A21})$$

and

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial \tilde{t}} + \tilde{\nabla} \times (\tilde{\boldsymbol{\omega}} \times \tilde{\mathbf{v}}) = -\tilde{\nabla} \tilde{p} \times \tilde{\nabla} \left(\frac{1}{\tilde{\rho}} \right). \quad (\text{A22})$$

A.4. Magnetic Fields

In the presence of magnetic fields, the momentum equation must be modified to take into account the magnetic tension and pressure. In the MHD approximation (where the fluid is assumed to have infinite conductivity), the momentum equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \Phi - \frac{\nabla p}{\rho} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi\rho}, \quad (\text{A23})$$

where \mathbf{B} is the magnetic field. The time evolution of the magnetic field is described by the following equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{c}{n_e^2 e} \nabla p_e \times \nabla n_e. \quad (\text{A24})$$

This is a standard MHD equation, except for the last term, which we have added to take into account the possibility of magnetic field generation in an ionized gas by a process known as the Biermann battery (Biermann 1950). In this term, p_e and n_e are the electron pressure and number density, respectively. If we assume that the ionized fraction χ is uniform and the electron, ion, and neutral temperatures are equal, this equation reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{cm_H}{e(1+\chi)} \nabla p \times \nabla \left(\frac{1}{\rho} \right), \quad (\text{A25})$$

(Kulsrud et al. 1997), where m_H is the hydrogen mass. To convert these equations to supercomoving variables, we consider two different definitions for the supercomoving magnetic field,

$$\tilde{\mathbf{B}}_1 = \frac{a^{5/2} \mathbf{B}}{B_*}, \quad (\text{A26})$$

$$\tilde{\mathbf{B}}_2 = \frac{a^2 \mathbf{B}}{B_*}, \quad (\text{A27})$$

where

$$B_* \equiv \rho_*^{1/2} v_*. \quad (\text{A28})$$

Using the definition (A26), the momentum equation becomes, in supercomoving variables,

$$\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + (\tilde{\mathbf{v}} \cdot \tilde{\nabla}) \tilde{\mathbf{v}} = -\frac{\tilde{\nabla} \tilde{p}}{\tilde{\rho}} - \tilde{\nabla} \tilde{\phi} + \frac{(\tilde{\nabla} \times \tilde{\mathbf{B}}_1) \times \tilde{\mathbf{B}}_1}{4\pi \tilde{\rho}}, \quad (\text{A29})$$

which has the same form as equation (A23), and equation (A24) becomes

$$\frac{\partial \tilde{\mathbf{B}}_1}{\partial \tilde{t}} - \frac{\mathcal{H}}{2} \tilde{\mathbf{B}}_1 = \tilde{\nabla} \times (\tilde{\mathbf{v}} \times \tilde{\mathbf{B}}_1) - \frac{a^{1/2}}{\rho_*^{1/2} r_*} \frac{cm_H}{e(1+\chi)} \tilde{\nabla} \tilde{p} \times \tilde{\nabla} \left(\frac{1}{\tilde{\rho}} \right). \quad (\text{A30})$$

Comparing equations (A24) and (A30), we see that the conversion to supercomoving variables introduces an “antidrag” term $\mathcal{H}\mathbf{B}/2$ (which also appears in standard comoving variables, though with a different coefficient), and an extra factor of $a^{1/2}$ in the Biermann term. Using the second definition (A27), instead, the momentum equation becomes

$$\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + (\tilde{\mathbf{v}} \cdot \tilde{\nabla}) \tilde{\mathbf{v}} = -\frac{\tilde{\nabla} \tilde{p}}{\tilde{\rho}} - \tilde{\nabla} \tilde{\phi} + \frac{a(\tilde{\nabla} \times \tilde{\mathbf{B}}_2) \times \tilde{\mathbf{B}}_2}{4\pi \tilde{\rho}}, \quad (\text{A31})$$

and equation (A24) becomes

$$\frac{\partial \tilde{\mathbf{B}}_2}{\partial \tilde{t}} = \tilde{\nabla} \times (\tilde{\mathbf{v}} \times \tilde{\mathbf{B}}_2) - \frac{1}{\rho_*^{1/2} r_*} \frac{cm_H}{e(1+\chi)} \tilde{\nabla} \tilde{p} \times \tilde{\nabla} \left(\frac{1}{\tilde{\rho}} \right), \quad (\text{A32})$$

which has a form very similar to equation (A24).

As noted recently by Kulsrud et al. (1997), there is a close correspondence between the inviscid equation for vorticity evolution, equation (A18) and the MHD equation for \mathbf{B} in the presence of the Biermann battery term, equation (A25), especially when \mathbf{B} is expressed in the same units as vorticity, by replacing \mathbf{B} by the cyclotron frequency $\omega_{\text{cyc}} = e\mathbf{B}/m_H c$. In the case in which both vorticity and magnetic field are initially zero, this leads to the remarkably simple result that

$$\omega_{\text{cyc}} = -\frac{\omega}{1+\chi}. \quad (\text{A33})$$

We can write equation (A32) in terms of $\tilde{\omega}_{\text{cyc}}$ defined by

$$\tilde{\omega}_{\text{cyc}} = \frac{\omega_{\text{cyc}} a^2}{\omega_{\text{cyc}*}}, \quad (\text{A34})$$

where

$$\omega_{\text{cyc}*} \equiv \frac{1}{t_*} = \omega_*. \quad (\text{A35})$$

This yields

$$\frac{\partial \tilde{\omega}_{\text{cyc}}}{\partial \tilde{t}} = \tilde{\nabla} \times (\tilde{\mathbf{v}} \times \tilde{\omega}_{\text{cyc}}) - \frac{1}{1+\chi} \tilde{\nabla} \tilde{p} \times \tilde{\nabla} \left(\frac{1}{\tilde{\rho}} \right). \quad (\text{A36})$$

A comparison of equations (A22) and (A36) then replaces equation (A33) by

$$\tilde{\omega}_{\text{cyc}} = -\frac{\tilde{\omega}}{1+\chi}. \quad (\text{A37})$$

Table 1. SUPERCOMOVING AGES OF THE UNIVERSE (PRESENT AND ASYMPTOTIC) FOR
FLAT MODELS WITH $n = 0$ AND $n = 1$.^a

Ω_0	$n = 0$			$n = 1$		
	a_0	\tilde{t}_0	\tilde{t}_∞	a_0	\tilde{t}_0	\tilde{t}_∞
0.01	4.6261	0.20324	0.21490	9.9499	0.27290	0.28350
0.02	3.6593	0.19630	0.21490	7.0000	0.26558	0.28350
0.05	2.6684	0.18015	0.21490	4.3589	0.24728	0.28350
0.10	2.0801	0.15834	0.21490	3.0000	0.22080	0.28350
0.20	1.5874	0.11014	0.21490	2.0000	0.17131	0.28350
0.50	1.0000	0.00000	0.21490	1.0000	0.00000	0.28350
0.80	0.6299	-0.21178	0.21490	0.5000	-0.33823	0.28350
0.90	0.4807	-0.38043	0.21490	0.3333	-0.63233	0.28350
0.99	0.2162	-1.07573	0.21490	0.1005	-2.02892	0.28350

^aPresent scale factor $a_0 \equiv a(\tilde{t}_0)$, where $\tilde{t}_0 = \tilde{t}(t_0)$, t_0 = present age of the universe in proper time, and $\tilde{t}_\infty = \tilde{t}(t = \infty)$, where t is proper time. We note that $\tilde{t}(t = 0) = -\infty$ in all cases.

Table 2. SELECTED FAMILIES OF INTEREST

Family	n	κ	Density Parameters
Einstein-de Sitter model	NA ^a	NA	$\Omega_0 = 1, \Omega_{X0} = 0$
Open, matter-dominated models	NA	NA	$\Omega_0 < 1, \Omega_{X0} = 0$
Closed, matter-dominated models	NA	NA	$\Omega_0 > 1, \Omega_{X0} = 0$
Flat models with massive neutrinos	3	NA	$\Omega_0 + \Omega_{X0} = 1$
Open models with massive neutrinos	3	NA	$\Omega_0 + \Omega_{X0} < 1$
Closed models with massive neutrinos	3	NA	$\Omega_0 + \Omega_{X0} > 1$
Marginally bound models with cosmic strings	2	NA	$\Omega_0 = 1$, any Ω_{X0}
Unbound models with cosmic strings	2	NA	$\Omega_0 < 1$, any Ω_{X0}
Bound models with cosmic strings	2	NA	$\Omega_0 > 1$, any Ω_{X0}
Flat models with a radiation background	4	0	$\Omega_0 + \Omega_{X0} = 1$
Flat models with a nonzero cosmological constant	0	0	$\Omega_0 + \Omega_{X0} = 1$
Critical models with a nonzero cosmological constant	0	$-3/4^{1/3}$	Equation (68)
Flat models with domain walls	1	0	$\Omega_0 + \Omega_{X0} = 1$
Critical models with domain walls	1	-2	Equation (68)

^aFor entries labeled “NA,” this means that the parameter is “not applicable.”

Figure Captions

Fig. 1.— Evolution of Cosmological Models in Supercomoving Variables. (a) (Top panel) Cosmic Scale Factor $a(\tilde{t})$ versus time \tilde{t} for various models. Solid curves (from left to right): open, flat, and closed, matter-dominated models. The solid dot indicates the location of the present for the flat model. Dotted curve: flat model with radiation. This curve merges with the solid curve for the flat matter-dominated model at late time. Short-dashed curves: critical models with a nonzero cosmological constant (lower curve) and with domain walls (upper curve). Long-dashed curves: flat models with a cosmological constant and with domain walls. (b) (Bottom panel) Supercomoving Hubble parameter $\mathcal{H} = a^{-1}da/d\tilde{t}$ versus \tilde{t} , for the same models as plotted in (a)

Fig. 2.— Families of Cosmological Models (a) (Top left panel) Family membership curves for $n = 0$. The curves are isocontours of the family membership parameter κ , as defined by equation (71), for particular values of κ , as labeled. The dashed curves represent particular cases mentioned in the text. For values in the shaded region, the universe does not begin with a Big Bang. (b) (Top right panel) Enlargement of lower right corner of (a). Models located below the dashed curve are bound, others are unbound. (c) (Middle left panel) Same as (a), except for models with $n = 1$. (d) (Middle right panel) Family Memberships for $n = 2$ models. There are 3 families, represented by the dashed line, and the 2 half-planes. (e) (Bottom left panel) Same as (d) except for models with $n = 3$. (f) (Bottom right) Same as (a) except for models with $n = 4$. Shaded area in (a) and (c) represents combination of parameters incompatible with the existence of a Big Bang.



